Chapter 3

Linear Transformations and Matrix Algebra

Section 3.1

Matrix Transformations

Motivation

Let A be a matrix, and consider the matrix equation $b = Ax$. If we vary x, we can think of this as a *function* of x.

Many functions in real life—the *linear* transformations—come from matrices in this way.

It makes us happy when a function comes from a matrix, because then questions about the function translate into questions a matrix, which we can usually answer.

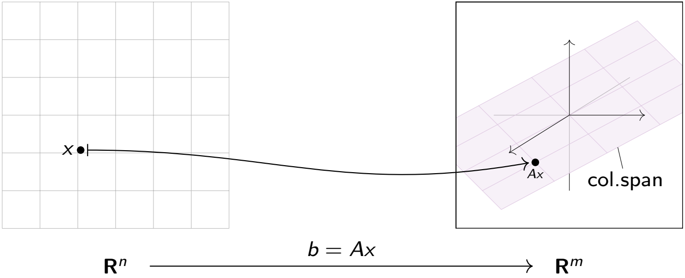
For this reason, we want to study matrices as functions.

Matrices as Functions

Change in Perspective. Let A be a matrix with m rows and n columns. Let's think about the matrix equation $b = Ax$ as a *function*.

- The independent variable (the input) is x, which is a vector in $\mathbf{R}^n$.
- The dependent variable (the output) is b, which is a vector in $\mathbf{R}^m$.

As you vary x, then $b = Ax$ also varies. The set of all possible output vectors b is the column space of A.



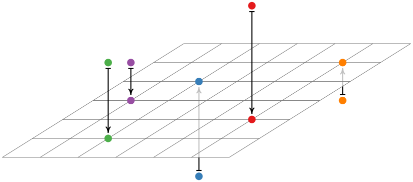
[interactive 1] [interactive 2]

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

In the equation $Ax = b$, the input vector x is in $\mathbf{R}^3$ and the output vector b is in $\mathbf{R}^3$. Then

$$A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}.$$

This is *projection onto the xy-plane*. Picture:



[interactive]

Matrices as Functions

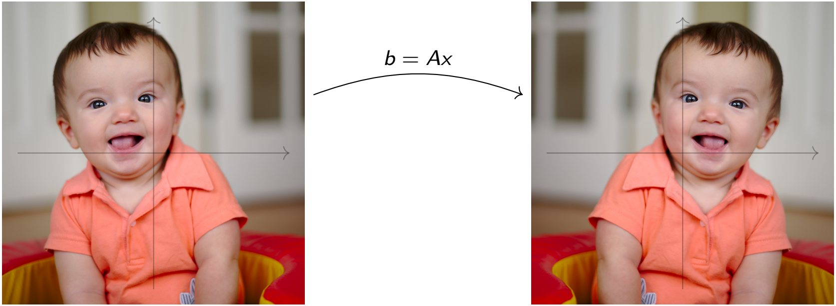
Reflection

$$A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

In the equation $Ax = b$, the input vector x is in $\mathbf{R}^2$ and the output vector b is in $\mathbf{R}^2$. Then

$$A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ y \end{pmatrix}.$$

This is *reflection over the y-axis*. Picture:



[interactive]

Matrices as Functions

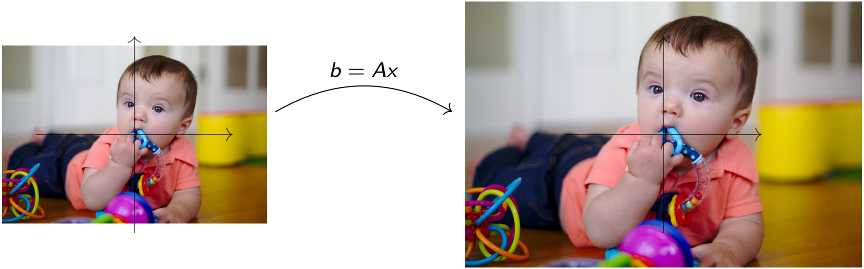
Dilation

$$A = \begin{pmatrix} 1.5 & 0 \\ 0 & 1.5 \end{pmatrix}$$

In the equation $Ax = b$, the input vector x is in $\mathbf{R}^2$ and the output vector b is in $\mathbf{R}^2$.

$$A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1.5 & 0 \\ 0 & 1.5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1.5x \\ 1.5y \end{pmatrix} = 1.5 \begin{pmatrix} x \\ y \end{pmatrix}.$$

This is *dilation (scaling) by a factor of 1.5*. Picture:



[interactive]

Matrices as Functions

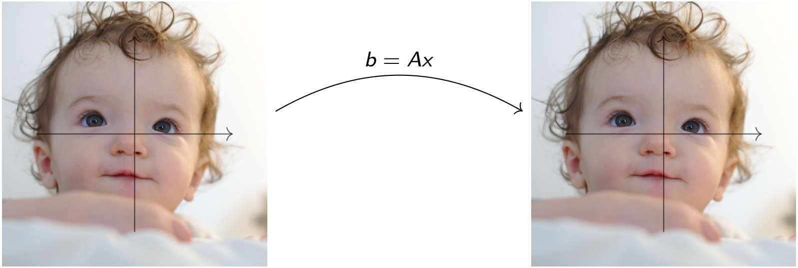
Identity

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

In the equation $Ax = b$, the input vector x is in $\mathbf{R}^2$ and the output vector b is in $\mathbf{R}^2$.

$$A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

This is *the identity transformation which does nothing*. Picture:



[interactive]

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

In the equation $Ax = b$, the input vector x is in $\mathbf{R}^2$ and the output vector b is in $\mathbf{R}^2$. Then

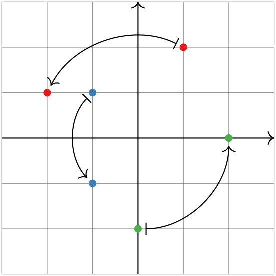
$$A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}.$$

What is this? Let's plug in a few points and see what happens.

$$A\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

$$A\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

$$A\begin{pmatrix} 0 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

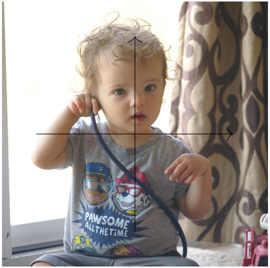


It looks like *counterclockwise rotation by* 90°.

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

In the equation $Ax = b$, the input vector x is in $\mathbf{R}^2$ and the output vector b is in $\mathbf{R}^2$. Then

$$A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}.$$



$\xrightarrow{b = Ax}$

[interactive]

Other Geometric Transformations

In §4.1 there are other examples of geometric transformations of $\mathbf{R}^2$ given by matrices. Please look them over.

We have been drawing pictures of what it looks like to multiply a matrix by a vector, as a function of the vector.

Now let's go the other direction. Suppose we have a function, and we want to know, does it come from a matrix?

Example

For a vector x in $\mathbf{R}^2$, let $T(x)$ be the counterclockwise rotation of x by an angle θ. Is $T(x) = Ax$ for some matrix A?

If $\theta = 90^\circ$, then we know $T(x) = Ax$, where

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

But for general θ, it's not clear.

Our next goal is to answer this kind of question.

Transformations

Vocabulary

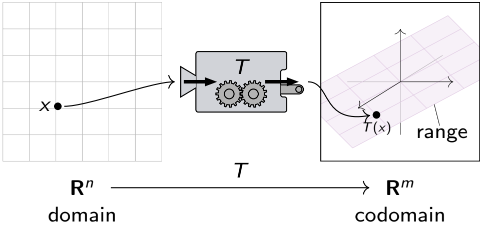
Definition

A **transformation** (or **function** or **map**) from $\mathbf{R}^n$ to $\mathbf{R}^m$ is a rule T that assigns to each vector x in $\mathbf{R}^n$ a vector $T(x)$ in $\mathbf{R}^m$.

- $\mathbf{R}^n$ is called the **domain** of T (the inputs).
- $\mathbf{R}^m$ is called the **codomain** of T (the outputs).
- For x in $\mathbf{R}^n$, the vector $T(x)$ in $\mathbf{R}^m$ is the **image** of x under T.
  Notation: $x \mapsto T(x)$.
- The set of all images $\{T(x) \mid x \text{ in } \mathbf{R}^n\}$ is the **range** of T.

Notation:

$$T\colon \mathbf{R}^n \longrightarrow \mathbf{R}^m \qquad \text{means} \qquad T \text{ is a transformation from } \mathbf{R}^n \text{ to } \mathbf{R}^m.$$



It may help to think of T as a "machine" that takes x as an input, and gives you $T(x)$ as the output.

Functions from Calculus

Many of the functions you know and love have domain and codomain $\mathbf{R}$.

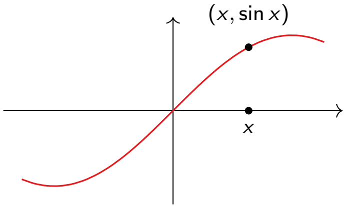
$$\sin\colon \mathbf{R} \longrightarrow \mathbf{R} \qquad \sin(x) = \left(\begin{array}{l}\text{the length of the opposite edge over the}\\ \text{hypotenuse of a right triangle with angle}\\ x \text{ in radians}\end{array}\right)$$

Note how I've written down the *rule* that defines the function sin.

$$f\colon \mathbf{R} \longrightarrow \mathbf{R} \qquad f(x) = x^2$$

Note that "x^2" is sloppy (but common) notation for a function: it doesn't have a name!

You may be used to thinking of a function in terms of its graph.

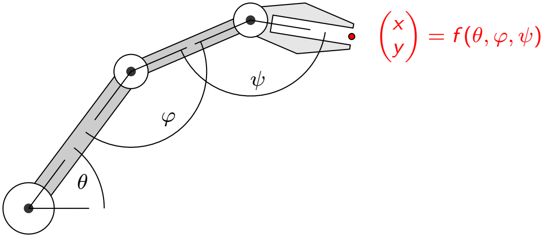


The horizontal axis is the domain, and the vertical axis is the codomain.

This is fine when the domain and codomain are $\mathbf{R}$, but it's hard to do when they're $\mathbf{R}^2$ and $\mathbf{R}^3$! You need five dimensions to draw that graph.

Functions from Engineering

Suppose you are building a robot arm with three joints that can move its hand around a plane, as in the following picture.



Define a transformation $f\colon \mathbf{R}^3 \to \mathbf{R}^2$:

$$f(\theta, \varphi, \psi) = \text{position of the hand at joint angles } \theta, \varphi, \psi.$$

Output of f: where is the hand on the plane.

This function does not come from a matrix; belongs to the field of inverse kinematics.

Matrix Transformations

Definition

Let A be an $m \times n$ matrix. The **matrix transformation** associated to A is the transformation

$$T\colon \mathbf{R}^n \longrightarrow \mathbf{R}^m \qquad \text{defined by} \qquad T(x) = Ax.$$

In other words, T takes the vector x in $\mathbf{R}^n$ to the vector Ax in $\mathbf{R}^m$.

For example, if $A = \left(\begin{smallmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{smallmatrix}\right)$ and $T(x) = Ax$ then

$$T\begin{pmatrix} -1 \\ -2 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}\begin{pmatrix} -1 \\ -2 \\ -3 \end{pmatrix} = \begin{pmatrix} -14 \\ -32 \end{pmatrix}.$$

Your life will be much easier if you just remember these.

- The *domain* of T is $\mathbf{R}^n$, which is the number of *columns* of A.
- The *codomain* of T is $\mathbf{R}^m$, which is the number of *rows* of A.
- The *range* of T is the set of all images of T:

$$T(x) = Ax = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x_1v_1 + x_2v_2 + \cdots + x_nv_n.$$

  This is the *column space* of A. It is a span of vectors in the codomain.

Matrix Transformations

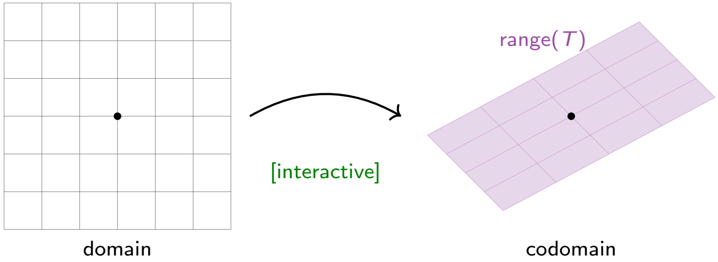
Example

$$A = \begin{pmatrix} -1 & 0 \\ 2 & 1 \\ 1 & -1 \end{pmatrix} \qquad T(x) = Ax \qquad T\colon \mathbf{R}^2 \to \mathbf{R}^3.$$

Domain is: $\mathbf{R}^2$. Codomain is: $\mathbf{R}^3$. Range is: all vectors of the form

$$T\begin{pmatrix} x \\ y \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix} = x\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix},$$

which is Col A.



Matrix Transformations

Picture

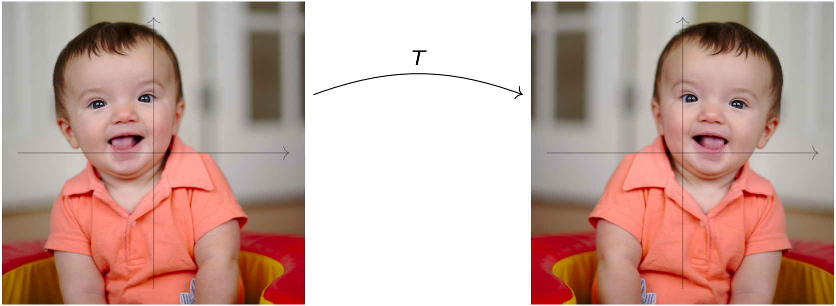
The picture of a matrix transformation is the same as the pictures we've been drawing all along. Only the language is different. Let

$$A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and let} \qquad T(x) = Ax,$$

so $T\colon \mathbf{R}^2 \to \mathbf{R}^2$. Then

$$T\begin{pmatrix} x \\ y \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ y \end{pmatrix},$$

which is still is *reflection over the y-axis*. Picture:

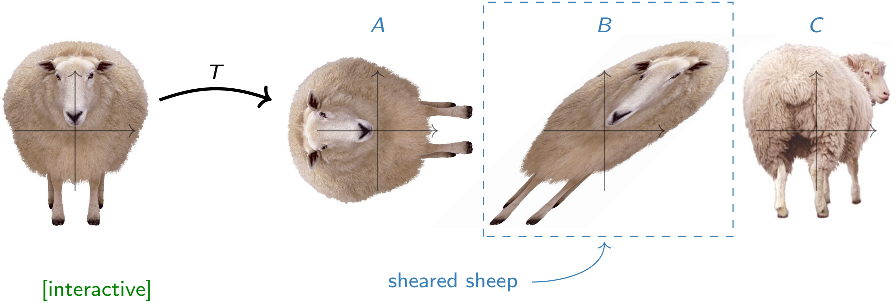


Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and let $T(x) = Ax$, so $T\colon \mathbf{R}^2 \to \mathbf{R}^2$. ($T$ is called a **shear**.)

Poll

What does T do to this sheep?

Hint: first draw a picture what it does to the box *around* the sheep.



[interactive]

Summary

- We can think of $b = Ax$ as a **transformation** with input x and output b.
- There are vocabulary words associated to transformations: **domain**, **codomain**, **range**.
- A transformation that comes from a matrix is called a **matrix transformation**.
- In this case, the vocabulary words mean something concrete in terms of matrices.
- We like transformations that come from matrices, because questions about those transformations turn into questions about matrices.