(Reduced) Row Echelon Form

Review from last time

A matrix is in **row echelon form** if

1. All zero rows are at the bottom.
2. Each leading nonzero entry of a row is to the *right* of the leading entry of the row above.
3. Below a leading entry of a row, all entries are *zero*.

A matrix is in **reduced row echelon form** if it is in row echelon form, and in addition,

4. The pivot in each nonzero row is equal to 1.
5. Each pivot is the only nonzero entry in its column.

Row echelon form:

$$\begin{pmatrix} \boxed{\star} & \star & \star & \star & \star \\ 0 & \boxed{\star} & \star & \star & \star \\ 0 & 0 & 0 & \boxed{\star} & \star \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Reduced row echelon form:

$$\begin{pmatrix} 1 & 0 & \star & 0 & \star \\ 0 & 1 & \star & 0 & \star \\ 0 & 0 & 0 & 1 & \star \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\boxed{\star}$ = **pivots**

Row Reduction: Theorem

Theorem

Every matrix is row equivalent to one and only one matrix in reduced row echelon form.

We'll give an algorithm, called **row reduction** or **Gaussian elimination**, which demonstrates that every matrix is row equivalent to *at least one* matrix in reduced row echelon form.

Note: Like echelon forms, the row reduction algorithm does not care if a column is augmented: ignore the vertical line when row reducing.

The uniqueness statement is interesting—it means that, nomatter *how* you row reduce, you *always* get the same matrix in reduced row echelon form. (Assuming you only do the three legal row operations.) (And you don't make any arithmetic errors.)

Maybe you can figure out why it's true!

Row Reduction Algorithm

Step 1a Swap the 1st row with a lower one so a leftmost nonzero entry is in 1st row (if necessary).

Step 1b Scale 1st row so that its leading entry is equal to 1.

Step 1c Use row replacement so all entries below this 1 are 0.

Step 2a Swap the 2nd row with a lower one so that the leftmost nonzero entry is in 2nd row.

Step 2b Scale 2nd row so that its leading entry is equal to 1.

Step 2c Use row replacement so all entries below this 1 are 0.

Step 3a Swap the 3rd row with a lower one so that the leftmost nonzero entry is in 3rd row.

etc.

Last Step Use row replacement to clear all entries above the pivots, starting with the last pivot (to make life easier).

Example

$$\left(\begin{array}{ccc|c} 0 & -7 & -4 & 2 \\ 2 & 4 & 6 & 12 \\ 3 & 1 & -1 & -2 \end{array}\right)$$

[animated]

$$\left(\begin{array}{ccc|c} \boxed{0} & -7 & -4 & 2 \\ 2 & 4 & 6 & 12 \\ 3 & 1 & -1 & -2 \end{array}\right) \xrightarrow{R_1 \longleftrightarrow R_2} \left(\begin{array}{ccc|c} \boxed{2} & 4 & 6 & 12 \\ 0 & -7 & -4 & 2 \\ 3 & 1 & -1 & -2 \end{array}\right)$$

Step 1a: Row swap to make this nonzero.

Step 1b: Scale to make this 1.

$$\xrightarrow{R_1 = R_1 \div 2} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ \boxed{3} & 1 & -1 & -2 \end{array}\right)$$

Step 1c: Subtract a multiple of the first row to clear this.

$$\xrightarrow{R_3 = R_3 - 3R_1} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 0 & -5 & -10 & -20 \end{array}\right)$$

Optional: swap rows 2 and 3 to make Step 2b easier later on.

$$\xrightarrow{R_2 \longleftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -5 & -10 & -20 \\ 0 & -7 & -4 & 2 \end{array}\right)$$

Row Reduction

Example, continued

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & \boxed{-5} & -10 & -20 \\ 0 & -7 & -4 & 2 \end{array}\right) \xrightarrow{R_2 = R_2 \div -5} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & \boxed{-7} & -4 & 2 \end{array}\right)$$

Step 2a: This is already nonzero.
Step 2b: Scale to make this 1.

(There are no fractions because of the optional step before.)

Step 2c: Add 7 times the second row to clear this.

$$\xrightarrow{R_3 = R_3 + 7R_2} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 10 & 30 \end{array}\right)$$

Note: Step 2 never messes up the first (nonzero) column of the matrix, because it looks like this:

$$\begin{array}{r} \\ \text{“Active” row} \rightarrow \\ \\ \end{array} \left(\begin{array}{ccc|c} 1 & \star & \star & \star \\ 0 & \star & \star & \star \\ 0 & \star & \star & \star \end{array}\right)$$

Row Reduction

Example, continued

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \boxed{10} & 30 \end{array}\right) \xrightarrow{R_3 = R_3 \div 10} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

Step 3a: This is already nonzero.
Step 3b: Scale to make this 1.

Note: Step 3 never messes up the columns to the left.
Note: The matrix is now in row echelon form!

$$\left(\begin{array}{ccc|c} 1 & 2 & \boxed{3} & 6 \\ 0 & 1 & \boxed{2} & 4 \\ 0 & 0 & 1 & 3 \end{array}\right) \xrightarrow{R_2 = R_2 - 2R_3} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

Last step: Add multiples of the third row to clear these.

$$\xrightarrow{R_1 = R_1 - 3R_3} \left(\begin{array}{ccc|c} 1 & \boxed{2} & 0 & -3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

Last step: Add -2 times the third row to clear this.

$$\xrightarrow{R_1 = R_1 - 2R_2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

Row Reduction
Example, continued

Success! The reduced row echelon form is

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right) \implies \left\{\begin{array}{rrrl} x & & & = 1 \\ & y & & = -2 \\ & & z & = 3 \end{array}\right.$$

Recap

Get a 1 here

$$\begin{pmatrix} \boxed{\star} & \star & \star & \star \\ \star & \star & \star & \star \\ \star & \star & \star & \star \\ \star & \star & \star & \star \end{pmatrix}$$

Clear down

$$\begin{pmatrix} \boxed{1} & \star & \star & \star \\ \star & \star & \star & \star \\ \star & \star & \star & \star \\ \star & \star & \star & \star \end{pmatrix}$$

Get a 1 here

$$\begin{pmatrix} 1 & \star & \star & \star \\ 0 & \boxed{\star} & \star & \star \\ 0 & \star & \star & \star \\ 0 & \star & \star & \star \end{pmatrix}$$

Clear down

$$\begin{pmatrix} 1 & \star & \star & \star \\ 0 & \boxed{1} & \star & \star \\ 0 & \star & \star & \star \\ 0 & \star & \star & \star \end{pmatrix}$$

(maybe these are already zero)

$$\begin{pmatrix} 1 & \star & \star & \star \\ 0 & 1 & \star & \star \\ 0 & 0 & 0 & \star \\ 0 & 0 & 0 & \star \end{pmatrix}$$

Get a 1 here

$$\begin{pmatrix} 1 & \star & \star & \star \\ 0 & 1 & \star & \star \\ 0 & 0 & 0 & \boxed{\star} \\ 0 & 0 & 0 & \star \end{pmatrix}$$

Clear down

$$\begin{pmatrix} 1 & \star & \star & \star \\ 0 & 1 & \star & \star \\ 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & \star \end{pmatrix}$$

Matrix is in REF

$$\begin{pmatrix} 1 & \star & \star & \star \\ 0 & 1 & \star & \star \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Clear up

$$\begin{pmatrix} 1 & \star & \star & \star \\ 0 & 1 & \star & \star \\ 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Clear up

$$\begin{pmatrix} 1 & \star & \star & 0 \\ 0 & \boxed{1} & \star & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Matrix is in RREF

$$\begin{pmatrix} 1 & 0 & \star & 0 \\ 0 & 1 & \star & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Profit?

The linear system

$$\begin{aligned} 2x + 10y &= -1 \\ 3x + 15y &= 2 \end{aligned} \qquad \text{gives rise to the matrix} \qquad \left(\begin{array}{cc|c} 2 & 10 & -1 \\ 3 & 15 & 2 \end{array}\right).$$

Let's row reduce it: [interactive row reducer]

$$\left(\begin{array}{cc|c} 2 & 10 & -1 \\ 3 & 15 & 2 \end{array}\right) \xrightarrow{R_1 = R_1 \div 2} \left(\begin{array}{cc|c} 1 & 5 & -\frac{1}{2} \\ 3 & 15 & 2 \end{array}\right) \qquad \text{(Step 1b)}$$

$$\xrightarrow{R_2 = R_2 - 3R_1} \left(\begin{array}{cc|c} 1 & 5 & -\frac{1}{2} \\ 0 & 0 & \frac{7}{2} \end{array}\right) \qquad \text{(Step 1c)}$$

$$\xrightarrow{R_2 = R_2 \times \frac{2}{7}} \left(\begin{array}{cc|c} 1 & 5 & -\frac{1}{2} \\ 0 & 0 & 1 \end{array}\right) \qquad \text{(Step 2b)}$$

$$\xrightarrow{R_1 = R_1 + \frac{1}{2}R_2} \left(\begin{array}{cc|c} 1 & 5 & 0 \\ 0 & 0 & 1 \end{array}\right) \qquad \text{(Step 2c)}$$

The row reduced matrix

$$\left(\begin{array}{cc|c} 1 & 5 & 0 \\ 0 & 0 & 1 \end{array}\right) \qquad \text{corresponds to the } \textit{inconsistent} \text{ system} \qquad \begin{aligned} x + 5y &= 0 \\ 0 &= 1. \end{aligned}$$

Inconsistent Matrices

Question

What does an augmented matrix in reduced row echelon form look like, if its system of linear equations is inconsistent?

Answer:

$$\left(\begin{array}{cccc|c} 1 & 0 & \star & \star & 0 \\ 0 & 1 & \star & \star & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

An augmented matrix corresponds to an inconsistent system of equations if and only if *the last* (i.e., the augmented) *column is a pivot column.*

Section 1.3

Parametric Form

Another Example

The linear system

$$\begin{aligned} 2x + y + 12z &= 1 \\ x + 2y + 9z &= -1 \end{aligned} \quad \text{gives rise to the matrix} \quad \left(\begin{array}{ccc|c} 2 & 1 & 12 & 1 \\ 1 & 2 & 9 & -1 \end{array}\right).$$

Let's row reduce it: [interactive row reducer]

$$\left(\begin{array}{ccc|c} 2 & 1 & 12 & 1 \\ 1 & 2 & 9 & -1 \end{array}\right) \xrightarrow{R_1 \longleftrightarrow R_2} \left(\begin{array}{ccc|c} 1 & 2 & 9 & -1 \\ 2 & 1 & 12 & 1 \end{array}\right) \quad \text{(Optional)}$$

$$\xrightarrow{R_2 = R_2 - 2R_1} \left(\begin{array}{ccc|c} 1 & 2 & 9 & -1 \\ 0 & -3 & -6 & 3 \end{array}\right) \quad \text{(Step 1c)}$$

$$\xrightarrow{R_2 = R_2 \div -3} \left(\begin{array}{ccc|c} 1 & 2 & 9 & -1 \\ 0 & 1 & 2 & -1 \end{array}\right) \quad \text{(Step 2b)}$$

$$\xrightarrow{R_1 = R_1 - 2R_2} \left(\begin{array}{ccc|c} 1 & 0 & 5 & 1 \\ 0 & 1 & 2 & -1 \end{array}\right) \quad \text{(Step 2c)}$$

The row reduced matrix

$$\left(\begin{array}{ccc|c} 1 & 0 & 5 & 1 \\ 0 & 1 & 2 & -1 \end{array}\right) \quad \text{corresponds to the linear system} \quad \begin{cases} x + 5z = 1 \\ y + 2z = -1 \end{cases}$$

Another Example

Continued

The system

$$\begin{aligned} x + 5z &= 1 \\ y + 2z &= -1 \end{aligned}$$

comes from a matrix in reduced row echelon form. Are we done? Is the system solved?

Yes! Rewrite:

$$\begin{aligned} x &= 1 - 5z \\ y &= -1 - 2z \end{aligned}$$

For any value of z, there is exactly one value of x and y that makes the equations true. But z can be *anything we want*!

So we have found the solution set: it is all values x, y, z where

$$\begin{aligned} x &= 1 - 5z \\ y &= -1 - 2z \\ (z &= z) \end{aligned} \qquad \text{for } z \text{ any real number.}$$

This is called the **parametric form** for the solution. [interactive picture]

For instance, $(1, -1, 0)$ and $(-4, -3, 1)$ are solutions.

Free Variables

Definition

Consider a *consistent* linear system of equations in the variables $x_1, \ldots, x_n$. Let A be a row echelon form of the matrix for this system.

We say that x_i is a **free variable** if its corresponding column in A is *not* a pivot column.

Important

1. You can choose *any value* for the free variables in a (consistent) linear system.
2. Free variables come from *columns without pivots* in a matrix in row echelon form.

In the previous example, z was free because the reduced row echelon form matrix was

$$\left(\begin{array}{ccc|c} 1 & 0 & 5 & 4 \\ 0 & 1 & 2 & -1 \end{array}\right).$$

In this matrix:

$$\left(\begin{array}{cccc|c} 1 & \star & 0 & \star & \star \\ 0 & 0 & 1 & \star & \star \end{array}\right)$$

the free variables are x_2 and x_4. (What about the last column?)

One More Example

The reduced row echelon form of the matrix for a linear system in x_1, x_2, x_3, x_4 is

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 3 & 2 \\ 0 & 0 & 1 & 4 & -1 \end{array}\right)$$

The free variables are x_2 and x_4: they are the ones whose columns are *not* pivot columns.

This translates into the system of equations

$$\left\{\begin{array}{rcrcr} x_1 & & + 3x_4 & = & 2 \\ & x_3 & + 4x_4 & = & -1 \end{array}\right. \qquad \Longrightarrow \qquad \boxed{\begin{array}{rcrcl} x_1 & = & 2 & - & 3x_4 \\ x_3 & = & -1 & - & 4x_4 \end{array}}$$

What happened to x_2? What is it allowed to be? Anything! The general solution is

$$(x_1,\, x_2,\, x_3,\, x_4) = (2 - 3x_4,\, x_2,\, -1 - 4x_4,\, x_4)$$

for any values of x_2 and x_4. For instance, $(2, 0, -1, 0)$ is a solution ($x_2 = x_4 = 0$), and $(5, 1, 3, -1)$ is a solution ($x_2 = 1$, $x_4 = -1$).

The boxed equation is called the **parametric form** of the general solution to the system of equations. It is obtained by moving all free variables to the right-hand side of the $=$.

Yet Another Example

The linear system

$$x + y + z = 1 \quad \text{has matrix form} \quad \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \end{array}\right).$$

This is in reduced row echelon form. The free variables are y and z. The parametric form of the general solution is

$$x = 1 - y - z.$$

Rearranging:

$$(x, y, z) = (1 - y - z,\ y,\ z),$$

where y and z are arbitrary real numbers. This was an example in the second lecture!

[interactive]

Poll

Poll

Is it possible for a system of linear equations to have exactly two solutions?

Trichotomy

There are *three possibilities* for the reduced row echelon form of the augmented matrix of a linear system.

1. The last column is a pivot column.
   In this case, the system is *inconsistent*. There are *zero* solutions, i.e. the solution set is *empty*. Picture:
$$\left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right)$$
2. Every column except the last column is a pivot column.
   In this case, the system has a *unique solution*. Picture:
$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & \star \\ 0 & 1 & 0 & \star \\ 0 & 0 & 1 & \star \end{array}\right)$$
3. The last column is not a pivot column, and some other column isn't either.
   In this case, the system has *infinitely many* solutions, corresponding to the infinitely many possible values of the free variable(s). Picture:
$$\left(\begin{array}{cccc|c} 1 & \star & 0 & \star & \star \\ 0 & 0 & 1 & \star & \star \end{array}\right)$$

Summary

- **Row reduction** is an algorithm for solving a system of linear equations represented by an augmented matrix.
- The goal of row reduction is to put a matrix into **(reduced) row echelon form**, which is the "solved" version of the matrix.
- An augmented matrix corresponds to an inconsistent system if and only if there is a pivot in the augmented column.
- Columns without pivots in the RREF of a matrix correspond to **free variables**. You can assign any value you want to the free variables, and you get a unique solution.
- A linear system has zero, one, or infinitely many solutions.