Math 1553 Worksheet §4.4 and 4.5, Matrix Multiplication and Inverses

Solutions

1. True or false (justify your answer). Answer true if the statement is *always* true. Otherwise, answer false.

a) If A is a 3×4 matrix and B is a 4×2 matrix, then the linear transformation transformation Z defined by $Z(x) = ABx$ has domain $\mathbf{R}^2$ and codomain $\mathbf{R}^3$.

b) If A is an $n \times n$ matrix and the equation $Ax = b$ has at least one solution for each b in $\mathbf{R}^n$, then the solution is *unique* for each b in $\mathbf{R}^n$.

c) Suppose A is an $n \times n$ matrix and every vector in $\mathbf{R}^n$ can be written as a linear combination of the columns of A. Then A must be invertible.

d) Suppose $T : \mathbf{R}^n \to \mathbf{R}^m$ and $U : \mathbf{R}^m \to \mathbf{R}^p$ are linear transformations and $U \circ T$ is onto. Then U and T must both be onto.

Solution.

a) True. In order for Bx to make sense, x must be in $\mathbf{R}^2$, and so Bx is in $\mathbf{R}^4$ and $A(Bx)$ is in $\mathbf{R}^3$. Therefore, the domain of Z is $\mathbf{R}^2$ and the codomain of Z is $\mathbf{R}^3$.

b) True. The first part says the transformation $T(x) = Ax$ is onto. Since A is $n \times n$, this is the same as saying A is invertible, so T is one-to-one and onto. Therefore, the equation $Ax = b$ has exactly one solution for each b in $\mathbf{R}^n$.

c) True. If the columns of A span $\mathbf{R}^n$, then A is invertible by the Invertible Matrix Theorem. We can also see this directly without quoting the IMT:

If the columns of A span $\mathbf{R}^n$, then A has n pivots, so A has a pivot in each row and column, hence its matrix transformation $T(x) = Ax$ is one-to-one and onto and thus invertible. Therefore, A is invertible.

d) False. Take the linear transformations $T : \mathbf{R}^3 \to \mathbf{R}^3$ and $U : \mathbf{R}^3 \to \mathbf{R}^2$ given by $T(x, y, z) = (x, y, 0)$ and $U(x, y, z) = (x, y)$. Then $(U \circ T)(x, y, z) = (x, y)$, so $U \circ T$ maps $\mathbf{R}^3$ onto $\mathbf{R}^2$. However, T is not onto since the z-coordinate of every vector in its image is 0.

2. Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be rotation *clockwise* by $60°$. Let $U : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation with standard matrix $\begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix}$.

a) Find the standard matrix for the composition $U \circ T$.

b) Find the standard matrix for the composition $T \circ U$.

c) Find the standard matrix for U^{-1}.

Solution.

a) The matrix for T is $\begin{pmatrix} \cos(-60°) & -\sin(-60°) \\ \sin(-60°) & \cos(-60°) \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$. The matrix for $U \circ T$ is

$$\begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} -1 - \frac{\sqrt{3}}{2} & \frac{1}{2} - \sqrt{3} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}.$$

b) The matrix for $T \circ U$ is

$$\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 + \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} + \sqrt{3} & -\frac{\sqrt{3}}{2} \end{pmatrix}.$$

c) The matrix for U^{-1} is B^{-1}. Recall that if $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\det(B) = ad - bc \neq 0$ then B is invertible and $B^{-1} = \frac{1}{\det(B)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

$$B^{-1} = \frac{1}{0-1} \begin{pmatrix} 0 & -1 \\ -1 & -2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}.$$