

Math 1553 Worksheet §4.2, 4.3

Solutions

1. Which of the following transformations T are onto? Which are one-to-one? If the transformation is not onto, find a vector not in the range. If the transformation is not one-to-one, find two vectors with the same image.
- a) Counterclockwise rotation by 32° in \mathbf{R}^2 .
 - b) The transformation $T : \mathbf{R}^3 \rightarrow \mathbf{R}^2$ defined by $T(x, y, z) = (z, x)$.
 - c) The transformation $T : \mathbf{R}^3 \rightarrow \mathbf{R}^2$ defined by $T(x, y, z) = (0, x)$.
 - d) The matrix transformation with standard matrix $A = \begin{pmatrix} 1 & 6 \\ -1 & 2 \\ 2 & -1 \end{pmatrix}$.
 - e) The matrix transformation with standard matrix $A = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

Solution.

- a) This is both one-to-one and onto. If v is any vector in \mathbf{R}^2 , then there is one and only one vector w such that $T(w) = v$, namely, the vector that is rotated -32° from v .
- b) This is onto. If (a, b) is any vector in the codomain \mathbf{R}^2 , then $(a, b) = T(b, 0, a)$, so (a, b) is in the range. It is not one-to-one though: indeed, $T(0, 0, 0) = (0, 0) = T(0, 1, 0)$. Alternatively, we could have observed that T is a matrix transformation and examined its matrix A : $T(x) = Ax$ for

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Since A has a pivot in every row but not every column, T is onto but not one-to-one.

- c) This is not onto. There is no (x, y, z) such that $T(x, y, z) = (1, 0)$. It is not one-to-one: for instance, $T(0, 0, 0) = (0, 0) = T(0, 1, 0)$.
- d) The transformation T with matrix A is onto if and only if A has a pivot in every row, and it is one-to-one if and only if A has a pivot in every column. So we row reduce:

$$A = \begin{pmatrix} 1 & 6 \\ -1 & 2 \\ 2 & -1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

This has a pivot in every column, so T is one-to-one. It does not have a pivot in every row, so it is not onto. To find a specific vector b in \mathbf{R}^3 which is not in the image of T , we have to find a $b = (b_1, b_2, b_3)$ such that the matrix equation

$Ax = b$ is inconsistent. We row reduce again:

$$(A | b) = \left(\begin{array}{cc|c} 1 & 6 & b_1 \\ -1 & 2 & b_2 \\ 2 & -1 & b_3 \end{array} \right) \xrightarrow{\text{rref}} \left(\begin{array}{cc|c} 1 & 0 & \text{don't care} \\ 0 & 1 & \text{don't care} \\ 0 & 0 & -3b_1 + 13b_2 + 8b_3 \end{array} \right).$$

Hence any b_1, b_2, b_3 for which $-3b_1 + 13b_2 + 8b_3 \neq 0$ will make the equation $Ax = b$ inconsistent. For instance, $b = (1, 0, 0)$ is not in the range of T .

- e) This matrix is already row reduced. We can see that does not have a pivot in every row or in every column, so it is neither onto nor one-to-one. In fact, if $T(x) = Ax$ then

$$T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 + 3x_2 \\ x_3 \\ 0 \end{pmatrix},$$

so we can see that $(0, 0, 1)$ is not in the range of T , and that

$$T \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = T \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}.$$

- On your computer, go to the [Interactive Transformation Challenge!](#) Complete the Zoom, Reflect, and Scale challenges. If you complete a challenge in the optimal number of steps, the interactive demo will congratulate you. See if you can complete each of these challenges in the optimal number of steps.
- The second little pig has decided to build his house out of sticks. His house is shaped like a pyramid with a triangular base that has vertices at the points $(0, 0, 0)$, $(2, 0, 0)$, $(0, 2, 0)$, and $(1, 1, 1)$.

The big bad wolf finds the pig's house and blows it down so that the house is rotated by an angle of 45° in a counterclockwise direction about the z -axis (look downward onto the xy -plane the way we usually picture the plane as \mathbf{R}^2), and then projected onto the xy -plane. Find the standard matrix A for the transformation T caused by the wolf.

Solution.

First notice that the little pig is a red herring, as it were—this is a question about the linear transformation T described in the last two lines.

To compute the matrix for T , we have to compute $T(e_1)$, $T(e_2)$, and $T(e_3)$. To see the picture, let's put ourselves above the xy -plane (with the usual orientation of the x and y axes in the xy -plane), looking downward. For e_1 and e_2 , it is as if we are applying $\begin{pmatrix} \cos(45^\circ) & -\sin(45^\circ) \\ \sin(45^\circ) & \cos(45^\circ) \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$ to $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, then

putting a zero in the z -coordinate each time. We find

$$T(e_1) = T \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad T(e_2) = T \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}.$$

Rotating e_3 around the z -axis does nothing, and projecting onto the xy -plane sends it to zero, so $T(e_3) = 0$. Therefore, the matrix for T is

$$A = \begin{pmatrix} | & | & | \\ T(e_1) & T(e_2) & T(e_3) \\ | & | & | \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$