Math 1553, Extra Practice for Midterm 3 (sections 5.1-6.6)

Solutions

1. In this problem, if the statement is always true, circle **T**; otherwise, circle **F**.

a) **T** **F** If A is row equivalent to B, then A and B have the same eigenvalues.

b) **T** **F** If A and B have the same eigenvectors, then A and B have the same characteristic polynomial.

c) **T** **F** If A is diagonalizable, then A has n distinct eigenvalues.

d) **T** **F** If A is an $n \times n$ matrix then $\det(-A) = -\det(A)$.

e) **T** **F** If A is an $n \times n$ matrix and its eigenvectors form a basis for $\mathbf{R}^n$, then A is invertible.

f) **T** **F** If 0 is an eigenvalue of the $n \times n$ matrix A, then $\operatorname{rank}(A) < n$.

Solution.

a) False: for instance, the matrices $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ are row equivalent, but have different eigenvalues.

b) False: $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ have the same eigenvectors (all nonzero vectors in $\mathbf{R}^2$) but characteristic polynomials λ^2 and $(1-\lambda)^2$, respectively.

c) False: for instance, $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is diagonal but has only one eigenvalue.

d) False: Since $\det(cA) = c^n \det(A)$, we see $\det(-A) = (-1)^n \det(A) = \det(A)$ if n is even.

e) False: False. For example, the zero matrix is not invertible but its eigenvectors form a basis for $\mathbf{R}^n$.

f) True: If $\lambda = 0$ is an eigenvalue of A then A is not invertible so its associated transformation $T(x) = Ax$ is not onto, hence $\operatorname{rank}(A) < n$.

2. In this problem, if the statement is always true, circle **T**; if it is always false, circle **F**; if it is sometimes true and sometimes false, circle **M**.

a) **T** **F** **M** If A is a 3×3 matrix with characteristic polynomial $-\lambda^3+\lambda^2+\lambda$, then A is invertible.

b) **T** **F** **M** A 3×3 matrix with (only) two distinct eigenvalues is diagonalizable.

c) **T** **F** **M** A diagonalizable $n\times n$ matrix admits n linearly independent eigenvectors.

d) **T** **F** **M** If $\det(A)=0$, then 0 is an eigenvalue of A.

Solution.

a) False: $\lambda=0$ is a root of the characteristic polynomial, so 0 is an eigenvalue, and A is not invertible.

b) Maybe: it is diagonalizable if and only if the eigenspace for the eigenvalue with multiplicity 2 has dimension 2.

c) True: by the Diagonalization Theorem, an $n\times n$ matrix is diagonalizable *if and only if* it admits n linearly independent eigenvectors.

d) True: if $\det(A)=0$ then A is not invertible, so $Av=0v$ has a nontrivial solution.

3. In this problem, you need not explain your answers; just circle the correct one(s).

Let A be an $n \times n$ matrix.

a) Which **one** of the following statements is correct?

1. An eigenvector of A is a vector v such that $Av = \lambda v$ for a nonzero scalar λ.
2. An eigenvector of A is a nonzero vector v such that $Av = \lambda v$ for a scalar λ.
3. An eigenvector of A is a nonzero scalar λ such that $Av = \lambda v$ for some vector v.
4. An eigenvector of A is a nonzero vector v such that $Av = \lambda v$ for a nonzero scalar λ.

b) Which **one** of the following statements is **not** correct?

1. An eigenvalue of A is a scalar λ such that $A - \lambda I$ is not invertible.
2. An eigenvalue of A is a scalar λ such that $(A - \lambda I)v = 0$ has a solution.
3. An eigenvalue of A is a scalar λ such that $Av = \lambda v$ for a nonzero vector v.
4. An eigenvalue of A is a scalar λ such that $\det(A - \lambda I) = 0$.

c) Which of the following 3×3 matrices are necessarily diagonalizable over the real numbers? (Circle all that apply.)

1. A matrix with three distinct real eigenvalues.
2. A matrix with one real eigenvalue.
3. A matrix with a real eigenvalue λ of algebraic multiplicity 2, such that the λ-eigenspace has dimension 2.
4. A matrix with a real eigenvalue λ such that the λ-eigenspace has dimension 2.

Solution.

a) Statement 2 is correct: an eigenvector must be nonzero, but its eigenvalue may be zero.

b) Statement 2 is incorrect: the solution v must be nontrivial.

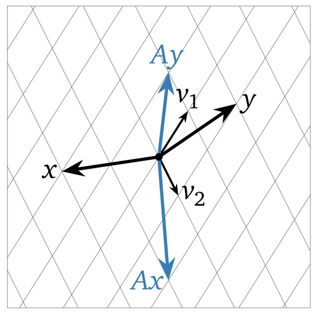
c) The matrices in 1 and 3 are diagonalizable. A matrix with three distinct real eigenvalues automatically admits three linearly independent eigenvectors. If a matrix A has a real eigenvalue λ_1 of algebraic multiplicity 2, then it has another real eigenvalue λ_2 of algebraic multiplicity 1. The two eigenspaces provide three linearly independent eigenvectors.

The matrices in 2 and 4 need not be diagonalizable.

4. Short answer.

a) Let $A = \begin{pmatrix} -1 & 1 \\ 1 & 7 \end{pmatrix}$, and define a transformation $T : \mathbf{R}^2 \to \mathbf{R}^2$ by $T(x) = Ax$. Find the area of $T(S)$, if S is a triangle in $\mathbf{R}^2$ with area 2.

b) Suppose that $A = C\begin{pmatrix} 1/2 & 0 \\ 0 & -1 \end{pmatrix}C^{-1}$, where C has columns v_1 and v_2. Given x and y in the picture below, draw the vectors Ax and Ay.



c) Write a diagonalizable 3×3 matrix A whose only eigenvalue is $\lambda = 2$.

Solution.

a) $|\det(A)|\text{Vol}(S) = |-7-1| \cdot 2 = 16.$

b) A does the same thing as $D = \begin{pmatrix} 1/2 & 0 \\ 0 & -1 \end{pmatrix}$, but in the v_1, v_2-coordinate system. Since D scales the first coordinate by $1/2$ and the second coordinate by -1, hence A scales the v_1-coordinate by $1/2$ and the v_2-coordinate by -1.

c) There is only one such matrix: $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

5. Suppose we know that

$$\begin{pmatrix} 4 & -10 \\ 2 & -5 \end{pmatrix} = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}^{-1}.$$

Find $\begin{pmatrix} 4 & -10 \\ 2 & -5 \end{pmatrix}^{98}$.

Solution.

$$\begin{pmatrix} 4 & -10 \\ 2 & -5 \end{pmatrix}^{98} = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}^{98}\begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ -2 & 5 \end{pmatrix}$$
$$= \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ -2 & 5 \end{pmatrix} = \begin{pmatrix} -4 & 10 \\ -2 & 5 \end{pmatrix}.$$

6. Let

$$A = \begin{pmatrix} 7 & 1 & 4 & 1 \\ -1 & 0 & 0 & 6 \\ 9 & 0 & 2 & 3 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 1 & 5 & 4 \\ 1 & -1 & -3 & 0 \\ -1 & 0 & 5 & 4 \\ 3 & -3 & -2 & 5 \end{pmatrix}$$

a) Compute $\det(A)$.

b) Compute $\det(B)$.

c) Compute $\det(AB)$.

d) Compute $\det(A^2B^{-1}AB^2)$.

Solution.

a) The second column has three zeros, so we expand by cofactors:

$$\det\begin{pmatrix} 7 & 1 & 4 & 1 \\ -1 & 0 & 0 & 6 \\ 9 & 0 & 2 & 3 \\ 0 & 0 & 0 & -1 \end{pmatrix} = -\det\begin{pmatrix} -1 & 0 & 6 \\ 9 & 2 & 3 \\ 0 & 0 & -1 \end{pmatrix}$$

Now we expand the second column by cofactors again:

$$\cdots = -2\det\begin{pmatrix} -1 & 6 \\ 0 & -1 \end{pmatrix} = (-2)(-1)(-1) = -2.$$

b) This is a complicated matrix without a lot of zeros, so we compute the determinant by row reduction. After one row swap and several row replacements, we reduce to the matrix

$$\begin{pmatrix} 1 & -1 & -3 & 0 \\ 0 & 1 & 5 & 4 \\ 0 & 0 & 7 & 8 \\ 0 & 0 & 0 & -3 \end{pmatrix}.$$

The determinant of this matrix is -21, so the determinant of the original matrix is 21.

c) $\det(AB) = \det(A)\det(B) = (-2)(21) = -42$.

d) $\det(A^2B^{-1}AB^2) = \det(A)^2\det(B)^{-1}\det(A)\det(B)^2 = \det(A)^3\det(B) = (-2)^3(21) = -168$.

7. Give an example of a 2×2 real matrix A with each of the following properties. You need not explain your answer.

a) A has no real eigenvalues.

b) A has eigenvalues 1 and 2.

c) A is diagonalizable but not invertible.

d) A is a rotation matrix with real eigenvalues.

Solution.

a) $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

b) $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$.

c) $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

d) $A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.

8. Consider the matrix

$$A = \begin{pmatrix} 4 & 2 & -4 \\ 0 & 2 & 0 \\ 2 & 2 & -2 \end{pmatrix}.$$

a) Find the eigenvalues of A, and compute their algebraic multiplicities.

b) For each eigenvalue of A, find a basis for the corresponding eigenspace.

c) Is A diagonalizable? If so, find an invertible matrix C and a diagonal matrix D such that $A = CDC^{-1}$. If not, why not?

Solution.

a) We compute the characteristic polynomial by expanding along the second row:

$$f(\lambda) = \det\begin{pmatrix} 4-\lambda & 2 & -4 \\ 0 & 2-\lambda & 0 \\ 2 & 2 & -2-\lambda \end{pmatrix} = (2-\lambda)\det\begin{pmatrix} 4-\lambda & -4 \\ 2 & -2-\lambda \end{pmatrix}$$
$$= (2-\lambda)(\lambda^2 - 2\lambda) = -\lambda(\lambda-2)^2$$

The roots are 0 (with multiplicity 1) and 2 (with multiplicity 2).

b) First we compute the 0-eigenspace by solving $(A - 0I)x = 0$:

$$A = \begin{pmatrix} 4 & 2 & -4 \\ 0 & 2 & 0 \\ 2 & 2 & -2 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The parametric vector form of the general solution is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = z\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, so a basis for the 0-eigenspace is $\left\{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\right\}$.

Next we compute the 2-eigenspace by solving $(A-2I)x = 0$:

$$A-2I = \begin{pmatrix} 2 & 2 & -4 \\ 0 & 0 & 0 \\ 2 & 2 & -4 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The parametric vector form for the general solution is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = y\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$, so a basis for the 2-eigenspace is $\left\{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}\right\}$.

c) We have produced three linearly independent eigenvectors, so the matrix is diagonalizable:

$$A = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}^{-1}.$$

9. Find all values of a so that $\lambda = 1$ an eigenvalue of the matrix A below.

$$A = \begin{pmatrix} 3 & -1 & 0 & a \\ a & 2 & 0 & 4 \\ 2 & 0 & 1 & -2 \\ 13 & a & -2 & -7 \end{pmatrix}$$

Solution.

We need to know which values of a make the matrix $A - I_4$ noninvertible. We have

$$A - I_4 = \begin{pmatrix} 2 & -1 & 0 & a \\ a & 1 & 0 & 4 \\ 2 & 0 & 0 & -2 \\ 13 & a & -2 & -8 \end{pmatrix}.$$

We expand cofactors along the third column, then the second column:

$$\begin{aligned}\det(A - I_4) &= 2\det\begin{pmatrix} 2 & -1 & a \\ a & 1 & 4 \\ 2 & 0 & -2 \end{pmatrix} \\ &= (2)(1)\det\begin{pmatrix} a & 4 \\ 2 & -2 \end{pmatrix} + (2)(1)\det\begin{pmatrix} 2 & a \\ 2 & -2 \end{pmatrix} \\ &= 2(-2a-8) + 2(-4-2a) = -8a - 24.\end{aligned}$$

This is zero if and only if $a = -3$.

10. Consider the matrix

$$A = \begin{pmatrix} 3\sqrt{3}-1 & -5\sqrt{3} \\ 2\sqrt{3} & -3\sqrt{3}-1 \end{pmatrix}$$

a) Find both complex eigenvalues of A.

b) Find an eigenvector corresponding to each eigenvalue.

Solution.

a) We compute the characteristic polynomial:

$$\begin{aligned} f(\lambda) &= \det\begin{pmatrix} 3\sqrt{3}-1-\lambda & -5\sqrt{3} \\ 2\sqrt{3} & -3\sqrt{3}-1-\lambda \end{pmatrix} \\ &= (-1-\lambda+3\sqrt{3})(-1-\lambda-3\sqrt{3}) + (2)(5)(3) \\ &= (-1-\lambda)^2 - 9(3) + 10(3) \\ &= \lambda^2 + 2\lambda + 4. \end{aligned}$$

By the quadratic formula,

$$\lambda = \frac{-2 \pm \sqrt{2^2 - 4(4)}}{2} = \frac{-2 \pm 2\sqrt{3}i}{2} = -1 \pm \sqrt{3}i.$$

b) Let $\lambda = -1 - \sqrt{3}i$. Then

$$A - \lambda I = \begin{pmatrix} (i+3)\sqrt{3} & -5\sqrt{3} \\ 2\sqrt{3} & (i-3)\sqrt{3} \end{pmatrix}.$$

Since $\det(A - \lambda I) = 0$, the second row is a multiple of the first, so a row echelon form of A is

$$\begin{pmatrix} i+3 & -5 \\ 0 & 0 \end{pmatrix}.$$

Hence an eigenvector with eigenvalue $-1-\sqrt{3}i$ is $v = \begin{pmatrix} 5 \\ 3+i \end{pmatrix}$. It follows that an eigenvector with eigenvalue $-1+\sqrt{3}i$ is $\overline{v} = \begin{pmatrix} 5 \\ 3-i \end{pmatrix}$.

11. The companies X, Y, and Z fight for customers. This year, company X has 40 customers, Company Y has 15 customers, and Z has 20 customers. Each year, the following changes occur:

- X keeps 75% of its customers, while losing 15% to Y and 10% to Z.
- Y keeps 60% of its customers, while losing 5% to X and 35% to Z.
- Z keeps 65% of its customers, while losing 15% to X and 20% to Y.

Write a stochastic matrix A and a vector x so that Ax will give the number of customers for firms X, Y, and Z (respectively) after one year. You do not need to compute Ax.

Solution.

$$A = \begin{pmatrix} 0.75 & 0.05 & 0.15 \\ 0.15 & 0.6 & 0.20 \\ 0.1 & 0.35 & 0.65 \end{pmatrix} \qquad x = \begin{pmatrix} 40 \\ 15 \\ 20 \end{pmatrix}.$$