MATH 1553, SPRING 2019
SAMPLE MIDTERM 2: 3.6 THROUGH 4.5

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Write your section number here: ____________

Please **read all instructions** carefully before beginning.

- The maximum score on this exam is 50 points.
- You have 50 minutes to complete this exam.
- There are no aids of any kind (calculators, notes, text, etc.) allowed.
- Please show your work unless instructed otherwise. A correct answer without appropriate work will receive little or no credit. If you cannot fit your work on the front side of the page, use the back side of the page as indicated.
- You may cite any theorem proved in class or in the sections we covered in the text.
- Good luck!

This is a practice exam. It is meant to be similar in format, length, and difficulty to the real exam. It is **not** meant as a comprehensive list of study problems. I recommend completing the practice exam in 50 minutes, without notes or distractions.

The exam is not designed to test material from the previous midterm on its own. However, knowledge of the material prior to section §3.6 is necessary for everything we do for the rest of the semester, so it is fair game for the exam as it applies to §§3.6 through 4.5.

Problem 1.

[2 points for each part]

Parts (a)-(d) are true or false. Circle **T** if the statement is *always* true. Otherwise, answer **F**. You do not need to justify your answer.

a) **T** **F** Suppose A is a matrix with more columns than rows. Then the matrix transformation $T(x) = Ax$ cannot be one-to-one.

b) **T** **F** If A is an $n \times n$ matrix and $Ax = b$ has exactly one solution for some b in $\mathbf{R}^n$, then A is invertible.

c) **T** **F** There are linear transformations $T : \mathbf{R}^4 \to \mathbf{R}^3$ and $U : \mathbf{R}^3 \to \mathbf{R}^4$ so that $T \circ U$ is onto.

d) **T** **F** $V = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \text{ in } \mathbf{R}^4 \;\middle|\; x - y = 1 + w \right\}$ is a subspace of $\mathbf{R}^4$.

e) Let $A = \begin{pmatrix} 2 & -1 \\ 2 & 1 \end{pmatrix}$. If A is invertible, find A^{-1}. If A is not invertible, justify why.

Solution.

a) True. A cannot have more pivots than rows, and since there are more columns than rows, this means A cannot have a pivot in every column.

b) True. If $Ax = b$ has exactly one solution for some b, then $Ax = 0$ has exactly one solution (since the sol. set for $Ax = b$ is a translate of the sol. set for $Ax = 0$), so A is invertible by the Invertible Matrix Theorem.

c) True. Take $U(x_1, x_2, x_3) = (x_1, x_2, x_3, 0)$ and $T(x_1, x_2, x_3, x_4) = (x_1, x_2, x_3)$. Then $(T \circ U)(x_1, x_2, x_3) = (x_1, x_2, x_3)$ so $T \circ U$ is onto (in fact, it is also one-to-one: it is the identity transformation!)

d) False: V doesn't contain the zero vector, so we see immediately that it is not a subspace of $\mathbf{R}^4$.

e) $\det(A) = 2 - (-2) = 4$, so A is invertible; $A^{-1} = \frac{1}{4}\begin{pmatrix} 1 & 1 \\ -2 & 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$.

Extra space for scratch work on problem 1

Problem 2.

[12 points]

Parts (a) to (d) are unrelated. You do not need to justify answers in (a) or (b).

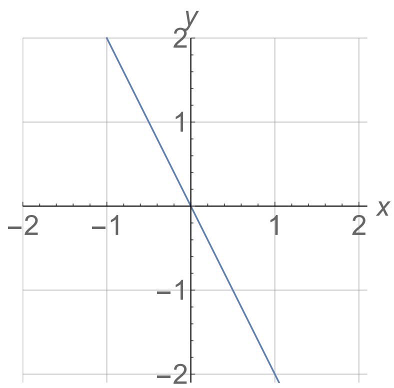
a) Write a matrix A whose null space is the line $y = x$ in $\mathbf{R}^2$ and whose matrix transformation $T(x) = Ax$ has range equal to the z-axis in $\mathbf{R}^3$.

b) Fill in the blanks: If A is a 5×6 matrix and its column span has dimension 2, then the null space of A is ________-dimensional subspace of $\mathbf{R}^{\square}$.

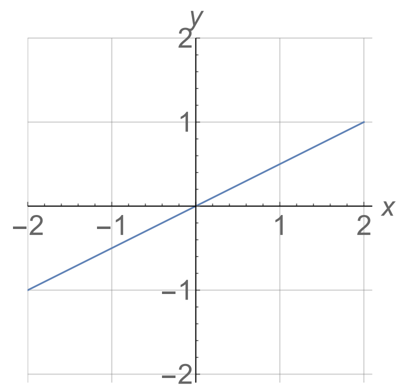
c) Let $A = \begin{pmatrix} 1 & 0 & -2 \\ 2 & 0 & -4 \\ -4 & 0 & 8 \end{pmatrix}$. Find nonzero vectors x and y in $\mathbf{R}^3$ so that $Ax = Ay$ but $x \neq y$.

d) Let $A = \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}$. Clearly draw Col($A$) and Nul($A$). Briefly show work.

Draw Col(A) here.



Draw Nul(A) here.



Solution.

a) Many examples possible. For example, $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & -1 \end{pmatrix}$ or $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ -4 & 4 \end{pmatrix}$.

b) Nul$A$ is a subspace of $\mathbf{R}^6$, and $\dim(\text{Col } A) + \dim(\text{Nul } A) = 6$ so $2 + \dim(\text{Nul}A) = 6$. Thus Nul A is a 4-dimensional subspace of $\mathbf{R}^6$.

c) We find two nonzero vectors in Nul$A$. $\left(A \mid 0\right) \to \left(\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$, so $x_1 = 2x_3$ and x_2 and x_3 are free. We can take $x = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ and $y = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, as $Ax = Ay = 0$.

d) Col(A) is the span of $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$; Nul($A$) is the span of $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

Extra space for work on problem 2

Problem 3. [10 points]

Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the transformation of reflection about the line $y = x$, and let $U : \mathbf{R}^2 \to \mathbf{R}^3$ be the transformation $U\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x-y \\ x \\ 3y \end{pmatrix}$.

a) Write the standard matrix A for T. Is T invertible?

b) Write the standard matrix B for U. Is U one-to-one? Briefly justify your answer.

c) Circle the composition that makes sense: $T \circ U$ $\qquad$ $U \circ T$

d) Compute the standard matrix for the composition you circled in part (c).

Solution.

a) $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Yes, T is invertible.

b) $B = \begin{pmatrix} U(e_1) & U(e_2) \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 0 & 3 \end{pmatrix}$. Yes, U is one-to-one because B has a pivot in every column.

c) $U \circ T$ makes sense since it sends $R^2 \to R^2 \to R^3$.

d) $BA = \begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 0 & 1 \\ 3 & 0 \end{pmatrix}$.

Extra space for work on problem 3

Problem 4.

[10 points]

McBarker has put the matrix A below in its reduced row echelon form:

$$A = \begin{pmatrix} 1 & -2 & 0 & 4 \\ -7 & 14 & 3 & 2 \\ 4 & -8 & -2 & -4 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & -2 & 0 & 4 \\ 0 & 0 & 1 & 10 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

a) Find a basis $\mathcal{B}$ for Nul(A).

b) Is $x = \begin{pmatrix} 2 \\ 3 \\ -10 \\ 1 \end{pmatrix}$ in Nul(A)? If so, write x as a linear combination of your basis vectors from part (a). If not, justify why x is not in Nul(A).

c) Is $\begin{pmatrix} 0 \\ -3 \\ 2 \end{pmatrix}$ in Col(A)? You do not need to justify your answer.

Solution.

a) The RREF of A shows that if $Ax = 0$ then

$$x_1 = 2x_2 - 4x_4,\ x_2 = x_2,\ x_3 = -10x_4,\ x_4 = x_4.$$

So $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2x_2 - 4x_4 \\ x_2 \\ -10x_4 \\ x_4 \end{pmatrix} = x_2 \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -4 \\ 0 \\ -10 \\ 1 \end{pmatrix}$. Thus $\mathcal{B} = \left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -4 \\ 0 \\ -10 \\ 1 \end{pmatrix} \right\}$.

b)

$$\left(\begin{array}{cc|c} 2 & -4 & 2 \\ 1 & 0 & 3 \\ 0 & -10 & -10 \\ 0 & 1 & 1 \end{array}\right) \xrightarrow[R_3 = -R_3/10]{R_1 \leftrightarrow R_2} \left(\begin{array}{cc|c} 1 & 0 & 3 \\ 2 & -4 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{array}\right) \xrightarrow[R_4 = R_4 - R_3]{R_2 = R_2 - 2R_1} \left(\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & -4 & -4 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right) \xrightarrow[R_3 = R_3 - R_2]{R_2 = R_2/-4} \left(\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right).$$

Therefore, x is in Nul(A), in fact $x = 3 \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -4 \\ 0 \\ -10 \\ 1 \end{pmatrix}$.

c) Yes. $\begin{pmatrix} 0 \\ -3 \\ 2 \end{pmatrix}$ is $(-1) * $ (third column of A) so it is in Col(A), no work required.

Extra space for work on problem 4

Problem 5. [8 points]

Parts (a), (b), and (c) are unrelated.
You do not need to show any work for parts (a) and (b).

a) Suppose that a linear transformation $T : \mathbf{R}^2 \to \mathbf{R}^2$ satisfies $T\begin{pmatrix}1\\-2\end{pmatrix} = \begin{pmatrix}1\\3\end{pmatrix}$ and $T\begin{pmatrix}3\\1\end{pmatrix} = \begin{pmatrix}-1\\1\end{pmatrix}$. Find $T\begin{pmatrix}4\\-1\end{pmatrix}$.

$\begin{pmatrix}4\\-1\end{pmatrix} = \begin{pmatrix}1\\-2\end{pmatrix} + \begin{pmatrix}3\\1\end{pmatrix}$, so by linearity of T,

$$T\begin{pmatrix}4\\-1\end{pmatrix} = T\begin{pmatrix}1\\-2\end{pmatrix} + T\begin{pmatrix}3\\1\end{pmatrix} = \begin{pmatrix}1\\3\end{pmatrix} + \begin{pmatrix}-1\\1\end{pmatrix} = \begin{pmatrix}0\\4\end{pmatrix}.$$

b) Give a specific example of a subspace of $\mathbf{R}^3$ that contains $\begin{pmatrix}1\\-1\\2\end{pmatrix}$.

(You may express this subspace any way you like, as long as you are clear.)

There are endless possibilities. For example, $\mathbf{R}^3$ itself, or Span $\left\{\begin{pmatrix}1\\-1\\2\end{pmatrix}\right\}$.

c) Write the invertible matrix A so that the transformation $T : \mathbf{R}^2 \to \mathbf{R}^2$ given by $T(x) = A^{-1}x$ reflects across the y-axis and then rotates counterclockwise by $\pi/2$ radians.

<u>Method 1</u>: One way is to find A^{-1} and calculate $A = (A^{-1})^{-1}$.

$$A^{-1} = \begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}\begin{pmatrix}-1 & 0\\0 & 1\end{pmatrix} = \begin{pmatrix}0 & -1\\-1 & 0\end{pmatrix}.$$

$$A = (A^{-1})^{-1} = \frac{1}{0-1}\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix} = \begin{pmatrix}0 & -1\\-1 & 0\end{pmatrix}.$$

(So, A is its own inverse!)

<u>Method 2</u>: Let B be the matrix for reflection across the y-axis and C be the matrix for rotation by $\pi/2$ radians counterclockwise. Then

$$A = (A^{-1})^{-1} = (CB)^{-1} = B^{-1}C^{-1}.$$

B is its own inverse since reflection across the y-axis twice gets you back where you started.
C's inverse is rotation clockwise by $\pi/2$ radians.

$$A = B^{-1}C^{-1} = \begin{pmatrix}-1 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix} = \begin{pmatrix}0 & -1\\-1 & 0\end{pmatrix}.$$

Extra space for work on problem 5