Math 1553 Supplement §6.5, 6.6

For those who want additional practice problems after completing the worksheet, here are some extra practice problems.

1. **a)** If A is the matrix that implements rotation by 143° in $\mathbf{R}^2$, then A has no real eigenvalues.

b) A 3×3 matrix can have eigenvalues $3, 5$, and $2+i$.

c) If $v = \begin{pmatrix} 2+i \\ 1 \end{pmatrix}$ is an eigenvector of A corresponding to the eigenvalue $\lambda = 1-i$, then $w = \begin{pmatrix} 2i-1 \\ i \end{pmatrix}$ is an eigenvector of A corresponding to the eigenvalue $\lambda = 1-i$.

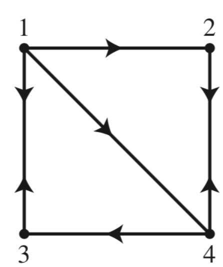
Solution.

a) True. If A had a real eigenvalue λ, then we would have $Ax = \lambda x$ for some nonzero vector x in $\mathbf{R}^2$. This means that x would lie on the same line through the origin as the rotation of x by 143°, which is impossible.

b) False. If $2+i$ is an eigenvalue then so is its conjugate $2-i$.

c) True. Any nonzero complex multiple of v is also an eigenvector for eigenvalue $1-i$, and $w = iv$.

2. Suppose the internet has four pages in the following manner. Arrows represent links from one page towards another. For example, page 1 links to page 4 but not vice versa.



a) Write the importance matrix and the Google matrix for this internet using damping constant $p = 0.15$. You don't need to simplify the Google matrix.

b) The steady-state vector for the Google matrix is (approximately)

$$\begin{pmatrix} 0.23 \\ 0.23 \\ 0.23 \\ 0.31 \end{pmatrix}.$$

What is the top-ranked page?

Solution.

(a) The importance matrix is

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1/3 & 0 & 0 & 1/2 \\ 1/3 & 0 & 0 & 1/2 \\ 1/3 & 1 & 0 & 0 \end{pmatrix}$$

The Google matrix is

$$(1-p)A + pB$$

$$0.85\begin{pmatrix} 0 & 0 & 1 & 0 \\ 1/3 & 0 & 0 & 1/2 \\ 1/3 & 0 & 0 & 1/2 \\ 1/3 & 1 & 0 & 0 \end{pmatrix} + (0.15)\frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

(b) From the steady-state vector we see page 4 has the highest rank.

3. Let $A = \begin{pmatrix} 4 & -3 & 3 \\ 3 & 4 & -2 \\ 0 & 0 & 2 \end{pmatrix}$. Find all eigenvalues of A. For each eigenvalue of A, find a corresponding eigenvector.

Solution.

First we compute the characteristic polynomial by expanding cofactors along the third row:

$$f(\lambda) = \det\begin{pmatrix} 4-\lambda & -3 & 3 \\ 3 & 4-\lambda & -2 \\ 0 & 0 & 2-\lambda \end{pmatrix} = (2-\lambda)\det\begin{pmatrix} 4-\lambda & -3 \\ 3 & 4-\lambda \end{pmatrix}$$
$$= (2-\lambda)\big((4-\lambda)^2 + 9\big) = (2-\lambda)(\lambda^2 - 8\lambda + 25).$$

Using the quadratic equation on the second factor, we find the eigenvalues

$$\lambda_1 = 2 \qquad \lambda_2 = 4 - 3i \qquad \overline{\lambda}_2 = 4 + 3i.$$

Next compute an eigenvector with eigenvalue $\lambda_1 = 2$:

$$A - 2I = \begin{pmatrix} 2 & -3 & 3 \\ 3 & 2 & -2 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

The parametric form is $x = 0$, $y = z$, so the parametric vector form of the solution is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = z\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \xrightarrow{\text{eigenvector}} v_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

Now we compute an eigenvector with eigenvalue $\lambda_2 = 4 - 3i$:

$$A = (4-3i)I = \begin{pmatrix} 3i & -3 & 3 \\ 3 & 3i & -2 \\ 0 & 0 & 3i-2 \end{pmatrix} \xrightarrow{R_1 \longleftrightarrow R_2} \begin{pmatrix} 3 & 3i & -2 \\ 3i & -3 & 3 \\ 0 & 0 & 3i-2 \end{pmatrix}$$

$$\xrightarrow{R_2 = R_2 - iR_1} \begin{pmatrix} 3 & 3i & -2 \\ 0 & 0 & 3+2i \\ 0 & 0 & 3i-2 \end{pmatrix} \xrightarrow{R_2 = R_2 \div (3+2i)} \begin{pmatrix} 3 & 3i & -2 \\ 0 & 0 & 1 \\ 0 & 0 & 3i-2 \end{pmatrix}$$

$$\xrightarrow{\text{row replacements}} \begin{pmatrix} 3 & 3i & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1 = R_1 \div 3} \begin{pmatrix} 1 & i & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

The parametric form of the solution is $x = -iy$, $z = 0$, so the parametric vector form is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = y \begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix} \xrightarrow{\text{eigenvector}} v_2 = \begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix}.$$

An eigenvector for the complex conjugate eigenvalue $\overline{\lambda}_2 = 4 + 3i$ is the complex conjugate eigenvector $\overline{v}_2 = \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix}$.