Math 1553 Supplement §6.2, 6.4

For those who want additional practice problems after completing the worksheet, here are some extra practice problems.

1. Give examples of 2×2 matrices with the following properties. Justify your answers.

a) A matrix A which is invertible and diagonalizable.

b) A matrix B which is invertible but not diagonalizable.

c) A matrix C which is not invertible but is diagonalizable.

d) A matrix D which is neither invertible nor diagonalizable.

Solution.

a) We can take any diagonal matrix with nonzero diagonal entries:

$$A=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.$$

b) A shear has only one eigenvalue $\lambda = 1$. The associated eigenspace is the x-axis, so there do not exist two linearly independent eigenvectors. Hence it is not diagonalizable.

$$B=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}.$$

c) We can take any diagonal matrix with some zero diagonal entries:

$$C=\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}.$$

d) Such a matrix can only have the eigenvalue zero — otherwise it would have two eigenvalues, hence be diagonalizable. Thus the characteristic polynomial is $f(\lambda)=\lambda^2$. Here is a matrix with trace and determinant zero, whose zero-eigenspace (i.e., null space) is not all of $\mathbf{R}^2$:

$$D=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}.$$

2. $A=\begin{pmatrix}2 & 3 & 1\\ 3 & 2 & 4\\ 0 & 0 & -1\end{pmatrix}.$

a) Find the eigenvalues of A, and find a basis for each eigenspace.

b) Is A diagonalizable? If your answer is yes, find a diagonal matrix D and an invertible matrix C so that $A = CDC^{-1}$. If your answer is no, justify why A is not diagonalizable.

Solution.

a) We solve $0 = \det(A - \lambda I)$.

$$0 = \det\begin{pmatrix} 2-\lambda & 3 & 1 \\ 3 & 2-\lambda & 4 \\ 0 & 0 & -1-\lambda \end{pmatrix} = (-1-\lambda)(-1)^6 \det\begin{pmatrix} 2-\lambda & 3 \\ 3 & 2-\lambda \end{pmatrix} = (-1-\lambda)((2-\lambda)^2 - 9)$$

$$= (-1-\lambda)(\lambda^2 - 4\lambda - 5) = -(\lambda+1)^2(\lambda-5).$$

So $\lambda = -1$ and $\lambda = 5$ are the eigenvalues.

$\underline{\lambda = -1}$: $$\left(A + I \mid 0\right) = \left(\begin{array}{ccc|c} 3 & 3 & 1 & 0 \\ 3 & 3 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_2 = R_2 - R_1} \left(\begin{array}{ccc|c} 3 & 3 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow[\text{then } R_1 = R_1/3]{R_1 = R_1 - R_2}$$

$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$, with solution $x_1 = -x_2$, $x_2 = x_2$, $x_3 = 0$. The (-1)-eigenspace has basis $\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right\}$.

$\underline{\lambda = 5}$:

$$\left(A - 5I \mid 0\right) = \left(\begin{array}{ccc|c} -3 & 3 & 1 & 0 \\ 3 & -3 & 4 & 0 \\ 0 & 0 & -6 & 0 \end{array}\right) \xrightarrow[R_3 = R_3/(-6)]{R_2 = R_2 + R_1} \left(\begin{array}{ccc|c} -3 & 3 & 1 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \xrightarrow[\text{then } R_2 \leftrightarrow R_3,\ R_1/(-3)]{R_1 = R_1 - R_3,\ R_2 = R_2 - 5R_3} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right),$$

with solution $x_1 = x_2,\ \ x_2 = x_2,\ \ x_3 = 0$. The 5-eigenspace has basis $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$.

b) A is a 3×3 matrix that only admits 2 linearly independent eigenvectors, so A is not diagonalizable.

3. Let $A = \begin{pmatrix} 8 & 36 & 62 \\ -6 & -34 & -62 \\ 3 & 18 & 33 \end{pmatrix}$.

The characteristic polynomial for A is $-\lambda^3 + 7\lambda^2 - 16\lambda + 12$, and $\lambda - 3$ is a factor. Decide if A is diagonalizable. If it is, find an invertible matrix C and a diagonal matrix D such that $A = CDC^{-1}$.

Solution.

By polynomial division,

$$\frac{-\lambda^3 + 7\lambda^2 - 16\lambda + 12}{\lambda - 3} = -\lambda^2 + 4\lambda - 4 = -(\lambda-2)^2.$$

Thus, the characteristic poly factors as $-(\lambda-3)(\lambda-2)^2$, so the eigenalues are $\lambda_1 = 3$ and $\lambda_2 = 2$.

For $\lambda_1 = 3$, we row-reduce $A - 3I$:

$$\begin{pmatrix} 5 & 36 & 62 \\ -6 & -37 & -62 \\ 3 & 18 & 30 \end{pmatrix} \xrightarrow[\text{(New } R_1)/3]{R_1 \leftrightarrow R_3} \begin{pmatrix} 1 & 6 & 10 \\ -6 & -37 & -62 \\ 5 & 36 & 62 \end{pmatrix} \xrightarrow[R_3 = R_3 - 5R_1]{R_2 = R_2 + 6R_1} \begin{pmatrix} 1 & 6 & 10 \\ 0 & -1 & -2 \\ 0 & 6 & 12 \end{pmatrix}$$

$$\xrightarrow[\text{then } R_2 = -R_2]{R_3 = R_3 + 6R_2} \begin{pmatrix} 1 & 6 & 10 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1 = R_1 - 6R_2} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

Therefore, the solutions to $\left(A - 3I \mid 0\right)$ are $x_1 = 2x_3, \ x_2 = -2x_3, \ x_3 = x_3$.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2x_3 \\ -2x_3 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}. \qquad \text{The 3-eigenspace has basis } \left\{ \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \right\}.$$

For $\lambda_2 = 2$, we row-reduce $A - 2I$:

$$\begin{pmatrix} 6 & 36 & 62 \\ -6 & -36 & -62 \\ 3 & 18 & 31 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 6 & \frac{31}{3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The solutions to $\left(A - 2I \quad 0\right)$ are $x_1 = -6x_2 - \frac{31}{3}x_3, \ x_2 = x_2, \ x_3 = x_3$.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -6x_2 - \frac{31}{3}x_3 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \begin{pmatrix} -6 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -\frac{31}{3} \\ 0 \\ 1 \end{pmatrix}.$$

The 2-eigenspace has basis $\left\{ \begin{pmatrix} -6 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{31}{3} \\ 0 \\ 1 \end{pmatrix} \right\}$.

Therefore, $A = CDC^{-1}$ where

$$C = \begin{pmatrix} 2 & -6 & -\frac{31}{3} \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Note that we arranged the eigenvectors in C in order of the eigenvalues $3, 2, 2$, so we had to put the diagonals of D in the same order.

4. Suppose a 2×2 matrix A has eigenvalue $\lambda_1 = -2$ with eigenvector $v_1 = \begin{pmatrix} 3/2 \\ 1 \end{pmatrix}$, and eigenvalue $\lambda_2 = -1$ with eigenvector $v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

a) Find A.

b) Find A^{100}.

Solution.

a) We have $A = CDC^{-1}$ where

$$C = \begin{pmatrix} 3/2 & 1 \\ 1 & -1 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}.$$

We compute $C^{-1} = \dfrac{1}{-5/2}\begin{pmatrix} -1 & -1 \\ -1 & 3/2 \end{pmatrix} = \dfrac{1}{5}\begin{pmatrix} 2 & 2 \\ 2 & -3 \end{pmatrix}$.

$$A = CDC^{-1} = \frac{1}{5}\begin{pmatrix} 3/2 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 2 & 2 \\ 2 & -3 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} -8 & -3 \\ -2 & -7 \end{pmatrix}.$$

b)

$$\begin{aligned} A^{100} = CD^{100}C^{-1} &= \frac{1}{5}\begin{pmatrix} 3/2 & 1 \\ 1 & -1 \end{pmatrix} \cdot D^{100}\begin{pmatrix} 2 & 2 \\ 2 & -3 \end{pmatrix} \\ &= \frac{1}{5}\begin{pmatrix} 3/2 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 2^{100} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 2 \\ 2 & -3 \end{pmatrix} \\ &= \frac{1}{5}\begin{pmatrix} 3/2 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 2 \cdot 2^{100} & 2 \cdot 2^{100} \\ 2 & -3 \end{pmatrix} \\ &= \frac{1}{5}\begin{pmatrix} 3 \cdot 2^{100} + 2 & 3 \cdot 2^{100} - 3 \\ 2^{101} - 2 & 2^{101} + 3 \end{pmatrix}. \end{aligned}$$