

## Math 1553 Supplement §4.2, 4.3

### Solutions

1. Let  $A$  be a  $3 \times 4$  matrix with column vectors  $v_1, v_2, v_3, v_4$ , and suppose  $v_2 = 2v_1 - 3v_4$ . Consider the matrix transformation  $T(x) = Ax$ .
- Is it possible that  $T$  is one-to-one? If yes, justify why. If no, find distinct vectors  $v$  and  $w$  so that  $T(v) = T(w)$ .
  - Is it possible that  $T$  is onto? Justify your answer.

### Solution.

- a) From the linear dependence condition we were given, we get

$$-2v_1 + v_2 + 3v_4 = 0.$$

The corresponding vector equation is just

$$(v_1 \ v_2 \ v_3 \ v_4) \begin{pmatrix} -2 \\ 1 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \text{so} \quad A \begin{pmatrix} -2 \\ 1 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Therefore,  $v = \begin{pmatrix} -2 \\ 1 \\ 0 \\ 3 \end{pmatrix}$  and  $w = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$  both satisfy  $Av = Aw = 0$ , so  $T$  cannot be one-to-one.

- b) Yes. If  $\{v_1, v_3, v_4\}$  is linearly independent then  $A$  will have a pivot in every row and  $T$  will be onto. Such a matrix  $A$  is

$$A = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -3 & 0 & 1 \end{pmatrix}.$$

2. Which of the following transformations  $T$  are onto? Which are one-to-one? If the transformation is not onto, find a vector not in the range. If the matrix is not one-to-one, find two vectors with the same image.
- The transformation  $T : \mathbf{R}^3 \rightarrow \mathbf{R}^2$  defined by  $T(x, y, z) = (y, y)$ .
  - JUST FOR FUN: Consider  $T : (\text{Smooth functions}) \rightarrow (\text{Smooth functions})$  given by  $T(f) = f'$  (the derivative of  $f$ ). Then  $T$  is not a transformation from any  $\mathbf{R}^n$  to  $\mathbf{R}^m$ , but it is still *linear* in the sense that for all smooth  $f$  and  $g$  and all scalars  $c$  (by properties of differentiation we learned in Calculus 1):

$$T(f + g) = T(f) + T(g) \quad \text{since} \quad (f + g)' = f' + g'$$

$$T(cf) = cT(f) \quad \text{since} \quad (cf)' = cf'.$$

Is  $T$  one-to-one?

**Solution.**

- a) This is not onto. Everything in the range of  $T$  has its first coordinate equal to its second, so there is no  $(x, y, z)$  such that  $T(x, y, z) = (1, 0)$ . It is not one-to-one: for instance,  $T(0, 0, 0) = (0, 0) = T(0, 0, 1)$ .
- b)  $T$  is not one-to-one. If  $T$  were one-to-one, then for any smooth function  $b$ , the equation  $T(f) = b$  would have at most one solution. However, Note that if  $f$  and  $g$  are the functions  $f(t) = t$  and  $g(t) = t - 1$ , then  $f$  and  $g$  are different functions but their derivatives are the same, so  $T(f) = T(g)$ . Therefore,  $T$  is not one-to-one. It is not within the scope of Math 1553. If you find it confusing, feel free to ignore it.
3. In each case, determine whether  $T$  is linear. Briefly justify.
- a)  $T(x_1, x_2) = (x_1 - x_2, x_1 + x_2, 1)$ .
- b)  $T(x, y) = (y, x^{1/3})$ .
- c)  $T(x, y, z) = 2x - 5z$ .

**Solution.**

- a) Not linear.  $T(0, 0) = (0, 0, 1) \neq (0, 0, 0)$ .
- b) Not linear. The  $x^{1/3}$  term gives it away.  $T(0, 2) = (0, 2^{1/3})$  but  $2T(0, 1) = (0, 2)$ .
- c) Linear. In fact,  $T(v) = Av$  where
- $$A = \begin{pmatrix} 2 & 0 & -5 \end{pmatrix}.$$
4. For each matrix  $A$ , describe what the associated matrix transformation  $T$  does to  $\mathbb{R}^3$  geometrically.

$$\text{a) } \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{b) } \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

**Solution.**

- a) We compute

$$T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ -y \\ z \end{pmatrix}.$$

This is the reflection over the  $xz$ -plane.

- b)

$$T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y \\ x \\ 0 \end{pmatrix}.$$

This is projection onto the  $xy$ -plane, followed by reflection over the line  $y = x$ .

5. Let's go back to the 4.2-4.3 worksheet problem #3. The second little pig has decided to build his house out of sticks. His house is shaped like a pyramid with a triangular base that has vertices at the points  $(0, 0, 0)$ ,  $(2, 0, 0)$ ,  $(0, 2, 0)$ , and  $(1, 1, 1)$ .

The big bad wolf finds the pig's house and blows it down so that the house is rotated by an angle of  $45^\circ$  in a counterclockwise direction about the  $z$ -axis (look downward onto the  $xy$ -plane the way we usually picture the plane as  $\mathbf{R}^2$ ), and then projected onto the  $xy$ -plane.

In the worksheet, we found the matrix for the transformation  $T$  caused by the wolf. Geometrically describe the image of the house under  $T$ .

### Solution.

In the worksheet, we found  $T(x) = Ax$  where

$$A = \begin{pmatrix} | & | & | \\ T(e_1) & T(e_2) & T(e_3) \\ | & | & | \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We know the house has been effectively destroyed, but what do its remains look like? To get an idea, let's look at what happens to the vertices.

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \sqrt{2} \\ \sqrt{2} \\ 0 \end{pmatrix}$$

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} -\sqrt{2} \\ \sqrt{2} \\ 0 \end{pmatrix}, \quad \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \sqrt{2} \\ 0 \end{pmatrix}.$$

This indicates the pyramid has been squashed into a triangle in the  $xy$ -plane with vertices  $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ ,  $\begin{pmatrix} \sqrt{2} \\ \sqrt{2} \\ 0 \end{pmatrix}$ ,  $\begin{pmatrix} -\sqrt{2} \\ \sqrt{2} \\ 0 \end{pmatrix}$ . (the point  $\begin{pmatrix} 0 \\ \sqrt{2} \\ 0 \end{pmatrix}$  is along the top side of this triangle).

Effectively, the pyramid was rotated and then destroyed, so that its (rotated) base is all that remains.