Math 1553 Supplement §3.6, 3.7, 3.9, 4.1

Solutions

1. Suppose V is a 3-dimensional subspace of $\mathbf{R}^5$ containing $\begin{pmatrix}1\\-4\\0\\0\\0\end{pmatrix}$, $\begin{pmatrix}1\\0\\-3\\1\\0\end{pmatrix}$, and $\begin{pmatrix}9\\8\\1\\0\\1\end{pmatrix}$.

Must $\left\{\begin{pmatrix}1\\-4\\0\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\-3\\1\\0\end{pmatrix}, \begin{pmatrix}9\\8\\1\\0\\1\end{pmatrix}\right\}$ be a basis for V? Justify your answer.

Solution.

Yes. The Basis Theorem says that since we know $\dim(V) = 3$, our three vectors will form a basis for V if and only if they are linearly independent.

Call the vectors v_1, v_2, v_3. It is very little work to show that the matrix $A = \begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix}$ has a pivot in every column, so the vectors are linearly independent.

2. Find bases for the column space and the null space of

$$A = \begin{pmatrix}0 & 1 & -3 & 1 & 0\\ 1 & -1 & 8 & -7 & 1\\ -1 & -2 & 1 & 4 & -1\end{pmatrix}.$$

Solution.

The RREF of $\left(A \mid 0\right)$ is

$$\left(\begin{array}{ccccc|c}1 & 0 & 5 & -6 & 1 & 0\\ 0 & 1 & -3 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\end{array}\right),$$

so x_3, x_4, x_5 are free, and

$$\begin{pmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{pmatrix} = \begin{pmatrix}-5x_3+6x_4-x_5\\ 3x_3-x_4\\ x_3\\ x_4\\ x_5\end{pmatrix} = x_3\begin{pmatrix}-5\\3\\1\\0\\0\end{pmatrix} + x_4\begin{pmatrix}6\\-1\\0\\1\\0\end{pmatrix} + x_5\begin{pmatrix}-1\\0\\0\\0\\1\end{pmatrix}.$$

Therefore, a basis for Nul A is $\left\{\begin{pmatrix}-5\\3\\1\\0\\0\end{pmatrix}, \begin{pmatrix}6\\-1\\0\\1\\0\end{pmatrix}, \begin{pmatrix}-1\\0\\0\\0\\1\end{pmatrix}\right\}$.

To find a basis for Col A, we use the pivot columns as they were written in the *original* matrix A, not its RREF. These are the first two columns:

$$\left\{\begin{pmatrix}0\\1\\-1\end{pmatrix},\begin{pmatrix}1\\-1\\-2\end{pmatrix}\right\}.$$

3. Find a basis for the subspace V of $\mathbf{R}^4$ given by

$$V=\left\{\begin{pmatrix}x\\y\\z\\w\end{pmatrix}\text{ in }\mathbf{R}^4 \;\middle|\; x+2y-3z+w=0\right\}.$$

Solution.

V is Nul A for the 1×4 matrix $A=\begin{pmatrix}1 & 2 & -3 & 1\end{pmatrix}$. The augmented matrix $\left(A \mid 0\right)=\left(\begin{array}{cccc|c}1 & 2 & -3 & 1 & 0\end{array}\right)$ gives $x=-2y+3z-w$ where y, z, w are free variables. The parametric vector form for the solution set to $Ax=0$ is

$$\begin{pmatrix}x\\y\\z\\w\end{pmatrix}=\begin{pmatrix}-2y+3z-w\\y\\z\\w\end{pmatrix}=y\begin{pmatrix}-2\\1\\0\\0\end{pmatrix}+z\begin{pmatrix}3\\0\\1\\0\end{pmatrix}+w\begin{pmatrix}-1\\0\\0\\1\end{pmatrix}.$$

Therefore, a basis for V is

$$\left\{\begin{pmatrix}-2\\1\\0\\0\end{pmatrix},\begin{pmatrix}3\\0\\1\\0\end{pmatrix},\begin{pmatrix}-1\\0\\0\\1\end{pmatrix}\right\}.$$

4. **a)** True or false: If A is an $m\times n$ matrix and $\mathrm{Nul}(A)=\mathbf{R}^n$, then $\mathrm{Col}(A)=\{0\}$.

b) Give an example of 2×2 matrix whose column space is the same as its null space.

c) True or false: For some m, we can find an $m\times 10$ matrix A whose column span has dimension 4 and whose solution set for $Ax=0$ has dimension 5.

Solution.

a) If $\mathrm{Nul}(A)=\mathbf{R}^n$ then $Ax=0$ for all x in $\mathbf{R}^n$, so the only element in $\mathrm{Col}(A)$ is $\{0\}$. Alternatively, the rank theorem says

$$\dim(\mathrm{Col}\ A)+\dim(\mathrm{Nul}\ A)=n \implies \dim(\mathrm{Col}\ A)+n=n \implies \dim(\mathrm{Col}\ A)=0 \implies \mathrm{Col}\ A=\{0\}.$$

b) Take $A=\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}$. Its null space and column space are $\mathrm{Span}\left\{\begin{pmatrix}1\\0\end{pmatrix}\right\}$.

c) False. The rank theorem says that the dimensions of the column space (Col$A$) and homogeneous solution space (Nul$A$) add to 10, no matter what m is.

5. For each matrix A, describe what the transformation $T(x) = Ax$ does to $\mathbf{R}^3$ geometrically.

$$\textbf{a)}\ \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \textbf{b)}\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Solution.

a) We compute

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -x \\ y \\ z \end{pmatrix}.$$

This is the reflection over the yz-plane.

b) We compute

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix}.$$

This is projection onto the z-axis.