Supplemental problems: §2.2, §2.3

Solutions

1. Put an augmented matrix into reduced row echelon form to solve the system

$$x_1 - 2x_2 - 9x_3 + x_4 = 3$$
$$4x_2 + 8x_3 - 24x_4 = 4.$$

Solution.

$$\left(\begin{array}{cccc|c} 1 & -2 & -9 & 1 & 3 \\ 0 & 4 & 8 & -24 & 4 \end{array}\right) \xrightarrow{R_2=\frac{R_2}{4}} \left(\begin{array}{cccc|c} 1 & -2 & -9 & 1 & 3 \\ 0 & 1 & 2 & -6 & 1 \end{array}\right) \xrightarrow{R_1=R_1+2R_2} \left(\begin{array}{cccc|c} \boxed{1} & 0 & -5 & -11 & 5 \\ 0 & \boxed{1} & 2 & -6 & 1 \end{array}\right)$$

The third and fourth columns are not pivot columns, so x_3 and x_4 are free variables. Our equations are

$$x_1 - 5x_3 - 11x_4 = 5$$
$$x_2 + 2x_3 - 6x_4 = 1.$$

Therefore,

$$\begin{aligned} x_1 &= 5 + 5x_3 + 11x_4 \\ x_2 &= 1 - 2x_3 + 6x_4 \\ x_3 &= x_3 \quad \text{(any real number)} \\ x_4 &= x_4 \quad \text{(any real number)} \end{aligned}$$

2. **a)** Row reduce the following matrices to reduced row echelon form.

b) If these are augmented matrices for a linear system (with the last column being after the = sign), then which are inconsistent? Which have a *unique* solution?

Solution.

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \\ 6 & 7 & 8 & 9 \end{pmatrix} \xrightarrow{R_2 = R_2 - 4R_1} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -6 & -9 \\ 6 & 7 & 8 & 9 \end{pmatrix}$$

$$\xrightarrow{R_3 = R_3 - 6R_1} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -6 & -9 \\ 0 & -5 & -10 & -15 \end{pmatrix}$$

$$\xrightarrow{R_2 = R_2 \div -3} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & -5 & -10 & -15 \end{pmatrix}$$

$$\xrightarrow{R_3 = R_3 + 5R_2} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{R_1 = R_1 - 2R_2} \begin{pmatrix} 1 & \color{red}{0} & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

This is the reduced row echelon form. Interpreted as an augmented matrix, it corresponds to the system of linear equations

$$\begin{aligned} x \quad - \;\; z &= -2 \\ y + 2z &= \;\;\, 3 \\ 0 &= \;\;\, 0. \end{aligned}$$

This system is consistent, but since z is a free variable, it does not have a *unique* solution.

$$\begin{pmatrix} 1 & 3 & 5 & 7 \\ 3 & 5 & 7 & 9 \\ 5 & 7 & 9 & 1 \end{pmatrix} \quad \xrightarrow{R_2 = R_2 - 3R_1} \begin{pmatrix} 1 & 3 & 5 & 7 \\ \color{red}{0} & -4 & -8 & -12 \\ 5 & 7 & 9 & 1 \end{pmatrix}$$

$$\xrightarrow{R_3 = R_3 - 5R_1} \begin{pmatrix} 1 & 3 & 5 & 7 \\ 0 & -4 & -8 & -12 \\ \color{red}{0} & -8 & -16 & -34 \end{pmatrix}$$

$$\xrightarrow{R_2 = R_2 \div -4} \begin{pmatrix} 1 & 3 & 5 & 7 \\ 0 & \color{red}{1} & 2 & 3 \\ 0 & -8 & -16 & -34 \end{pmatrix}$$

$$\xrightarrow{R_3 = R_3 + 8R_2} \begin{pmatrix} 1 & 3 & 5 & 7 \\ 0 & 1 & 2 & 3 \\ 0 & \color{red}{0} & 0 & -10 \end{pmatrix}$$

$$\xrightarrow{R_3 = R_3 \div -10} \begin{pmatrix} 1 & 3 & 5 & 7 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & \color{red}{1} \end{pmatrix}$$

$$\xrightarrow{R_1 = R_1 - 7R_3} \begin{pmatrix} 1 & 3 & 5 & \color{red}{0} \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{R_2 = R_2 - 3R_3} \begin{pmatrix} 1 & 3 & 5 & 0 \\ 0 & 1 & 2 & \color{red}{0} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{R_1 = R_1 - 3R_2} \begin{pmatrix} 1 & \color{red}{0} & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

This is the reduced row echelon form. Interpreted as an augmented matrix, it corresponds to the system of linear equations

$$\begin{aligned} x \quad - \;\; z &= 0 \\ y + 2z &= 0 \\ 0 &= 1, \end{aligned}$$

which is inconsistent.

$$
\begin{pmatrix} 3 & -4 & 2 & 0 \\ -8 & 12 & -4 & 0 \\ -6 & 8 & -1 & 0 \end{pmatrix}
\xrightarrow{R_2 = R_2 + 3R_1}
\begin{pmatrix} 3 & -4 & 2 & 0 \\ 1 & 0 & 2 & 0 \\ -6 & 8 & -1 & 0 \end{pmatrix}
$$

$$
\xrightarrow{R_1 \longleftrightarrow R_2}
\begin{pmatrix} 1 & 0 & 2 & 0 \\ 3 & -4 & 2 & 0 \\ -6 & 8 & -1 & 0 \end{pmatrix}
$$

$$
\xrightarrow{R_2 = R_2 - 3R_1}
\begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & -4 & -4 & 0 \\ -6 & 8 & -1 & 0 \end{pmatrix}
$$

$$
\xrightarrow{R_3 = R_3 + 6R_1}
\begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & -4 & -4 & 0 \\ 0 & 8 & 11 & 0 \end{pmatrix}
$$

$$
\xrightarrow{R_2 = R_2 \div -4}
\begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 8 & 11 & 0 \end{pmatrix}
$$

$$
\xrightarrow{R_3 = R_3 - 8R_2}
\begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 3 & 0 \end{pmatrix}
$$

$$
\xrightarrow{R_3 = R_3 \div 3}
\begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}
$$

$$
\xrightarrow{R_1 = R_1 - 2R_3}
\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}
$$

$$
\xrightarrow{R_2 = R_2 - R_3}
\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}
$$

This is the reduced row echelon form. Interpreted as an augmented matrix, it corresponds to the system of linear equations

$$x = 0 \quad y = 0 \quad z = 0,$$

which has a unique solution.

3. We can use linear algebra to find a polynomial that fits given data, in the same way that we found a circle through three specified points in the §2.1 Webwork.

Is there a degree-three polynomial $P(x)$ whose graph passes through the points $(-2,6)$, $(-1,4)$, $(1,6)$, and $(2,22)$? If so, how many degree-three polynomials have a graph through those four points? We answer this question in steps below.

a) If $P(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$ is a degree-three polynomial passing through the four points listed above, then $P(-2) = 6$, $P(-1) = 4$, $P(1) = 6$, and $P(2) = 22$. Write a system of four equations which we would solve to find a_0, a_1, a_2, and a_3.

b) Write the augmented matrix to represent this system and put it into reduced row-echelon form. Is the system consistent? How many solutions does it have?

Solution.

a) We compute

$$\begin{aligned} P(-2)=6 &\implies & a_0+a_1\cdot(-2)+a_2\cdot(-2)^2+a_3\cdot(-2)^3=6,\\ P(-1)=4 &\implies & a_0+a_1\cdot(-1)+a_2\cdot(-1)^2+a_3\cdot(-1)^3=4,\\ P(1)=6 &\implies & a_0+a_1\cdot 1+a_2\cdot 1^2+a_3\cdot 1^3=6,\\ P(2)=22 &\implies & a_0+a_1\cdot 2+a_2\cdot 2^2+a_3\cdot 2^3=22. \end{aligned}$$

Simplifying gives us

$$\begin{aligned} a_0-2a_1+4a_2-8a_3&=6\\ a_0-a_1+a_2-a_3&=4\\ a_0+a_1+a_2+a_3&=6\\ a_0+2a_1+4a_2+8a_3&=22. \end{aligned}$$

b) The corresponding augmented matrix is

$$\left(\begin{array}{cccc|c} 1 & -2 & 4 & -8 & 6\\ 1 & -1 & 1 & -1 & 4\\ 1 & 1 & 1 & 1 & 6\\ 1 & 2 & 4 & 8 & 22 \end{array}\right)$$

We label pivots with boxes as we proceed along. First, we subtract row 1 from each of rows 2, 3, and 4.

$$\left(\begin{array}{cccc|c} \boxed{1} & -2 & 4 & -8 & 6\\ 1 & -1 & 1 & -1 & 4\\ 1 & 1 & 1 & 1 & 6\\ 1 & 2 & 4 & 8 & 22 \end{array}\right) \rightsquigarrow \left(\begin{array}{cccc|c} \boxed{1} & -2 & 4 & -8 & 6\\ 0 & \boxed{1} & -3 & 7 & -2\\ 0 & 3 & -3 & 9 & 0\\ 0 & 4 & 0 & 16 & 16 \end{array}\right)$$

We now create zeros below the second pivot by subtracting multiples of the second row, then divide by 6 to get

$$\left(\begin{array}{cccc|c} \boxed{1} & -2 & 4 & -8 & 6\\ 0 & \boxed{1} & -3 & 7 & -2\\ 0 & 0 & \boxed{6} & -12 & 6\\ 0 & 0 & 12 & -12 & 24 \end{array}\right) \xrightarrow{R_3=R_3\div 6} \left(\begin{array}{cccc|c} \boxed{1} & -2 & 4 & -8 & 6\\ 0 & \boxed{1} & -3 & 7 & -2\\ 0 & 0 & \boxed{1} & -2 & 1\\ 0 & 0 & 12 & -12 & 24 \end{array}\right).$$

Now we subtract a 12 times row 3 from row 4 and divide by 12:

$$\left(\begin{array}{cccc|c} \boxed{1} & -2 & 4 & -8 & 6\\ 0 & \boxed{1} & -3 & 7 & -2\\ 0 & 0 & \boxed{1} & -2 & 1\\ 0 & 0 & 0 & \boxed{12} & 12 \end{array}\right) \xrightarrow{R_4=R_4\div 12} \left(\begin{array}{cccc|c} \boxed{1} & -2 & 4 & -8 & 6\\ 0 & \boxed{1} & -3 & 7 & -2\\ 0 & 0 & \boxed{1} & -2 & 1\\ 0 & 0 & 0 & \boxed{1} & 1 \end{array}\right).$$

At this point we can actually use back-substitution to solve: the last row says $a_3 = 1$, then plugging in $a_3 = 1$ in the third row gives us $a_2 = 3$, etc. However, for the sake of practice with reduced echelon form, let's keep row-reducing.

From right to left, we create zeros above the pivots in pivot columns by subtracting multiples of the pivot columns.

$$\left(\begin{array}{cccc|c} \boxed{1} & -2 & 4 & -8 & 6 \\ 0 & \boxed{1} & -3 & 7 & -2 \\ 0 & 0 & \boxed{1} & -2 & 1 \\ 0 & 0 & 0 & \boxed{1} & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{cccc|c} \boxed{1} & -2 & 4 & 0 & 14 \\ 0 & \boxed{1} & -3 & 0 & -9 \\ 0 & 0 & \boxed{1} & 0 & 3 \\ 0 & 0 & 0 & \boxed{1} & 1 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{cccc|c} \boxed{1} & -2 & 0 & 0 & 2 \\ 0 & \boxed{1} & 0 & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 & 3 \\ 0 & 0 & 0 & \boxed{1} & 1 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{cccc|c} \boxed{1} & 0 & 0 & 0 & 2 \\ 0 & \boxed{1} & 0 & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 & 3 \\ 0 & 0 & 0 & \boxed{1} & 1 \end{array}\right)$$

So $a_0 = 2$, $a_1 = 0$, $a_2 = 3$, and $a_3 = 1$. In other words,

$$\boxed{P(x) = 2 + 3x^2 + x^3}.$$

Therefore, there is exactly one third-degree polynomial satisfying the conditions of the problem. (You should check that, in fact, we have $P(-2) = 6$, $P(-1) = 4$, etc.)