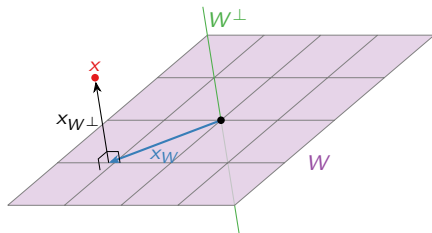


Orthogonal Projections

Review

Recall: Let W be a subspace of \mathbf{R}^n .

- ▶ The **orthogonal complement** W^\perp is the set of vectors orthogonal to everything in W .
- ▶ The **orthogonal decomposition** of a vector x with respect to W is the unique way of writing $x = x_W + x_{W^\perp}$ for x_W in W and x_{W^\perp} in W^\perp .
- ▶ The vector x_W is the **orthogonal projection** of x onto W . It is the closest vector to x in W .
- ▶ To compute x_W , write W as $\text{Col } A$ and solve $A^T A v = A^T x$; then $x_W = A v$.



Projection as a Transformation

Change in Perspective: let us consider orthogonal projection as a *transformation*.

Definition

Let W be a subspace of \mathbf{R}^n . Define a transformation

$$T: \mathbf{R}^n \longrightarrow \mathbf{R}^n \quad \text{by} \quad T(x) = x_W.$$

This transformation is also called **orthogonal projection** with respect to W .

Theorem

Let W be a subspace of \mathbf{R}^n and let $T: \mathbf{R}^n \rightarrow \mathbf{R}^n$ be the orthogonal projection with respect to W . Then:

1. T is a *linear* transformation.
2. For every x in W , $T(x)$ is the *closest* vector to x in W .
3. For every x in W , we have $T(x) = x$.
4. For every x in W^\perp , we have $T(x) = 0$.
5. $T \circ T = T$.
6. The range of T is W and the null space of T is W^\perp .

Projection Matrix

Method 1

Let W be a subspace of \mathbf{R}^n and let $T: \mathbf{R}^n \rightarrow \mathbf{R}^n$ be the orthogonal projection with respect to W .

Since T is a linear transformation, it has a matrix. How do you compute it?

The same as any other linear transformation: compute $T(e_1), T(e_2), \dots, T(e_n)$.

Projection Matrix

Example

Problem: Let $L = \text{Span}\left\{\begin{pmatrix} 3 \\ 2 \end{pmatrix}\right\}$ and let $T: \mathbf{R}^2 \rightarrow \mathbf{R}^2$ be the orthogonal projection onto L . Compute the matrix A for T .

It's easy to compute orthogonal projection onto a line:

$$\left. \begin{aligned} T(\mathbf{e}_1) &= (\mathbf{e}_1)_L = \frac{\mathbf{u} \cdot \mathbf{e}_1}{\mathbf{u} \cdot \mathbf{u}} \mathbf{u} = \frac{3}{13} \begin{pmatrix} 3 \\ 2 \end{pmatrix} \\ T(\mathbf{e}_2) &= (\mathbf{e}_2)_L = \frac{\mathbf{u} \cdot \mathbf{e}_2}{\mathbf{u} \cdot \mathbf{u}} \mathbf{u} = \frac{2}{13} \begin{pmatrix} 3 \\ 2 \end{pmatrix} \end{aligned} \right\} \implies A = \frac{1}{13} \begin{pmatrix} 9 & 6 \\ 6 & 4 \end{pmatrix}.$$

Projection Matrix

Another Example

Problem: Let

$$W = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \text{ in } \mathbf{R}^3 \mid x_1 - x_2 + x_3 = 0 \right\}$$

and let $T: \mathbf{R}^3 \rightarrow \mathbf{R}^3$ be orthogonal projection onto W . Compute the matrix B for T .

In the slides for the last lecture we computed $W = \text{Col } A$ for

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

To compute $T(e_i)$ we have to solve the matrix equation $A^T A v = A^T e_i$. We have

$$A^T A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \quad A^T e_i = \text{the } i\text{th column of } A^T = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}.$$

Projection Matrix

Another Example, Continued

Problem: Let

$$W = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \text{ in } \mathbf{R}^3 \mid x_1 - x_2 + x_3 = 0 \right\}$$

and let $T: \mathbf{R}^3 \rightarrow \mathbf{R}^3$ be orthogonal projection onto W . Compute the matrix B for T .

$$\left(\begin{array}{cc|c} 2 & -1 & 1 \\ -1 & 2 & -1 \end{array} \right) \xrightarrow{\text{RREF}} \left(\begin{array}{cc|c} 1 & 0 & 1/3 \\ 0 & 1 & -1/3 \end{array} \right) \implies T(e_1) = \frac{1}{3}A \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 2 & -1 & 1 \\ -1 & 2 & 0 \end{array} \right) \xrightarrow{\text{RREF}} \left(\begin{array}{cc|c} 1 & 0 & 2/3 \\ 0 & 1 & 1/3 \end{array} \right) \implies T(e_2) = \frac{1}{3}A \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 2 & -1 & 0 \\ -1 & 2 & 1 \end{array} \right) \xrightarrow{\text{RREF}} \left(\begin{array}{cc|c} 1 & 0 & 1/3 \\ 0 & 1 & 2/3 \end{array} \right) \implies T(e_3) = \frac{1}{3}A \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$$

$$\implies A = \frac{1}{3} \begin{pmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{pmatrix}.$$

Projection Matrix

Method 2

Theorem

Let $\{v_1, v_2, \dots, v_m\}$ be a *linearly independent* set in \mathbf{R}^n , and let

$$A = \begin{pmatrix} | & | & \cdots & | \\ v_1 & v_2 & \cdots & v_m \\ | & | & \cdots & | \end{pmatrix}.$$

Then the $m \times m$ matrix $A^T A$ is invertible.

Proof: We'll show $\text{Nul}(A^T A) = \{0\}$. Suppose $A^T A v = 0$. Then Av is in $\text{Nul}(A^T) = \text{Col}(A)^\perp$. But Av is in $\text{Col}(A)$ as well, so $Av = 0$, and hence $v = 0$ because the columns of A are linearly independent.

Projection Matrix

Method 2

Theorem

Let $\{v_1, v_2, \dots, v_m\}$ be a *linearly independent* set in \mathbf{R}^n , and let

$$A = \begin{pmatrix} | & | & \cdots & | \\ v_1 & v_2 & \cdots & v_m \\ | & | & \cdots & | \end{pmatrix}.$$

Then the $m \times m$ matrix $A^T A$ is invertible.

Let W be a subspace of \mathbf{R}^n and let $T: \mathbf{R}^n \rightarrow \mathbf{R}^n$ be the orthogonal projection with respect to W . Let $\{v_1, v_2, \dots, v_m\}$ be a *basis* for W and let A be the matrix with columns v_1, v_2, \dots, v_m . To compute $T(x) = x_W$ you solve $A^T A v = A^T x$; then $x_W = Av$.

$$v = (A^T A)^{-1} (A^T x) \implies T(x) = Av = [A(A^T A)^{-1} A^T] x.$$

If the columns of A are a *basis* for W then the matrix for T is

$$A(A^T A)^{-1} A^T.$$

Projection Matrix

Example

Problem: Let $L = \text{Span}\left\{\begin{pmatrix} 3 \\ 2 \end{pmatrix}\right\}$ and let $T: \mathbf{R}^2 \rightarrow \mathbf{R}^2$ be the orthogonal projection onto L . Compute the matrix A for T .

The set $\left\{\begin{pmatrix} 3 \\ 2 \end{pmatrix}\right\}$ is a basis for L , so

$$A = u(u^T u)^{-1} u^T = \frac{1}{u \cdot u} uu^T = \frac{1}{13} \begin{pmatrix} 3 \\ 2 \end{pmatrix} \begin{pmatrix} 3 & 2 \end{pmatrix} = \frac{1}{13} \begin{pmatrix} 9 & 6 \\ 6 & 4 \end{pmatrix}.$$

Matrix of Projection onto a Line

If $L = \text{Span}\{u\}$ is a line in \mathbf{R}^n , then the matrix for projection onto L is

$$\frac{1}{u \cdot u} uu^T.$$

Projection Matrix

Another Example

Problem: Let

$$W = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \text{ in } \mathbf{R}^3 \mid x_1 - x_2 + x_3 = 0 \right\}$$

and let $T: \mathbf{R}^3 \rightarrow \mathbf{R}^3$ be orthogonal projection onto W . Compute the matrix B for T .

In the slides for the last lecture we computed $W = \text{Col } A$ for

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The columns are linearly independent, so they form a basis for W . Hence

$$\begin{aligned} B &= A(A^T A)^{-1} A^T = A \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}^{-1} A^T = \frac{1}{3} A \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} A^T \\ &= \frac{1}{3} \begin{pmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{pmatrix}. \end{aligned}$$

Let W be a subspace of \mathbf{R}^n .

Poll

Let A be the matrix for proj_W . What are the possible eigenvalues of A ?

A. 0 B. 1 C. -1 D. 0, 1 E. 1, -1 F. 0, -1 G. -1, 0, 1

A destroys everything in W^\perp and fixes everything in W , so possible eigenvalues are 0 and 1.

We have $\dim W + \dim W^\perp = n$, so that gives n linearly independent eigenvectors already.

So the answer is D.

Projection Matrix

Facts

Theorem

Let W be an m -dimensional subspace of \mathbf{R}^n , let $T: \mathbf{R}^n \rightarrow W$ be the projection, and let A be the matrix for T . Then:

1. $\text{Col } A = W$, which is the 1-eigenspace.
2. $\text{Nul } A = W^\perp$, which is the 0-eigenspace.
3. $A^2 = A$.
4. A is similar to the diagonal matrix with m ones and $n - m$ zeros on the diagonal.

Proof of 4: Let v_1, v_2, \dots, v_m be a basis for W , and let $v_{m+1}, v_{m+2}, \dots, v_n$ be a basis for W^\perp . These are (linearly independent) eigenvectors with eigenvalues 1 and 0, respectively, and they form a basis for \mathbf{R}^n because there are n of them.

Example: If W is a plane in \mathbf{R}^3 , then A is similar to projection onto the xy -plane:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Computing the Projection Matrix by Diagonalizing

Let W be an m -dimensional subspace of \mathbf{R}^n , let $T: \mathbf{R}^n \rightarrow \mathbf{R}^n$ be the orthogonal projection onto W , and let A be the matrix for T . Here's how to compute A :

- ▶ Find a basis $\{v_1, v_2, \dots, v_m\}$ for W .
- ▶ Find a basis $\{v_{m+1}, v_{m+2}, \dots, v_n\}$ for W^\perp .
- ▶ Then

$$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix} \begin{matrix} \overbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}}^{m \text{ ones, } n-m \text{ zeros}} \end{matrix} \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix}^{-1}$$

Remark: If you already have a basis for W , then it's faster to compute $A(A^T A)^{-1} A^T$.

Projection Matrix

Example

Problem: Let

$$W = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \text{ in } \mathbf{R}^3 \mid x_1 - x_2 + x_3 = 0 \right\}$$

and let $T: \mathbf{R}^3 \rightarrow \mathbf{R}^3$ be orthogonal projection onto W . Compute the matrix B for T .

As we have seen several times, a basis for W is

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

By definition, W is the orthogonal complement of the line spanned by $(1, -1, 1)$, so $W^\perp = \text{Span}\{(1, -1, 1)\}$. Hence

$$B = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}^{-1} = \frac{1}{3} \begin{pmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{pmatrix}.$$

Let W be a subspace of \mathbf{R}^n and let x be a vector in \mathbf{R}^n .

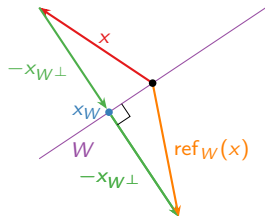
Definition

The **reflection** of x over W is the vector $\text{ref}_W(x) = x - 2x_{W^\perp}$.

In other words, to find $\text{ref}_W(x)$ one starts at x , then moves to $x - x_{W^\perp} = x_W$, then continues in the same direction one more time, to end on the opposite side of W .

Since $x_{W^\perp} = x - x_W$ we have

$$\text{ref}_W(x) = x - 2(x - x_W) = 2x_W - x.$$



If T is the orthogonal projection, then

$$\text{ref}_W(x) = 2T(x) - x.$$

Reflections

Properties

Theorem

Let W be an m -dimensional subspace of \mathbf{R}^n , and let A be the matrix for ref_W . Then

1. $\text{ref}_W \circ \text{ref}_W$ is the identity transformation and A^2 is the identity matrix.
2. ref_W and A are invertible; they are their own inverses.
3. The 1-eigenspace of A is W and the -1 -eigenspace of A is W^\perp .
4. A is similar to the diagonal matrix with m ones and $n - m$ negative ones on the diagonal.
5. If B is the matrix for the orthogonal projection onto W , then $A = 2B - I_n$.

Example: Let

$$W = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \text{ in } \mathbf{R}^3 \mid x_1 - x_2 + x_3 = 0 \right\}.$$

The matrix for ref_W is

$$A = 2 \cdot \frac{1}{3} \begin{pmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{pmatrix} - I_3 = \frac{1}{3} \begin{pmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ -2 & 2 & 1 \end{pmatrix}.$$

Today we considered orthogonal projection as a transformation.

- ▶ Orthogonal projection is a linear transformation.
- ▶ We gave three methods to compute its matrix.
- ▶ Four if you count the special case when W is a line.
- ▶ The matrix for projection onto W has eigenvalues 1 and 0 with eigenspaces W and W^\perp .
- ▶ A projection matrix is diagonalizable.
- ▶ Reflection is $2 \times$ projection minus the identity.