

# Chapter 5

## Determinants


# Section 5.1

Determinants: Definition

**Recall:** This course is about learning to:

- ▶ Solve the matrix equation  $Ax = b$   
We've said most of what we'll say about this topic now.
- ▶ Solve the matrix equation  $Ax = \lambda x$  (eigenvalue problem)  
We are now aiming at this.
- ▶ Almost solve the equation  $Ax = b$   
This will happen later.

The next topic is *determinants*.

This is a completely magical  function that takes a square matrix and gives you a number.

It is a very complicated function—the formula for the determinant of a  $10 \times 10$  matrix has 3,628,800 summands—so instead of writing down the formula, we'll give other ways to compute it.

Today is mostly about the *theory* of the determinant; in the next lecture we will focus on *computation*.

# A Definition of Determinant

## Definition

The **determinant** is a function

determinants are only for square matrices!

$$\det: \{n \times n \text{ matrices}\} \rightarrow \mathbf{R}$$

with the following properties:

1. If you do a row replacement on a matrix, the determinant doesn't change.
2. If you scale a row by  $c$ , the determinant is multiplied by  $c$ .
3. If you swap two rows of a matrix, the determinant is multiplied by  $-1$ .
4.  $\det(I_n) = 1$ .

Example:

$$\begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 1 & 4 \\ 2 & 1 \end{pmatrix}$$

$$\xrightarrow{R_2 = R_2 - 2R_1} \begin{pmatrix} 1 & 4 \\ 0 & -7 \end{pmatrix}$$

$$\xrightarrow{R_2 = R_2 \div -7} \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$$

$$\xrightarrow{R_1 = R_1 - 4R_2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$\det = 7$  ✓

$\det = -7$

$\det = 1$

$\det = 1$

## A Definition of Determinant

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4.  $\det(I_n) = 1$ .

This is a *definition* because it tells you how to compute the determinant: row reduce!

It's not at all obvious that you get the same determinant if you row reduce in two different ways, but this is magically true!

## Special Case 1

If  $A$  has a zero row, then  $\det(A) = 0$ .

Why?

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 7 & 8 & 9 \end{pmatrix} \xrightarrow{R_2 = -R_2} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 7 & 8 & 9 \end{pmatrix}$$

The determinant of the second matrix is negative the determinant of the first (property 3), so

$$\det \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 7 & 8 & 9 \end{pmatrix} = -\det \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 7 & 8 & 9 \end{pmatrix}.$$

This implies the determinant is zero.

## Special Case 2

If  $A$  is upper-triangular, then the determinant is the product of the diagonal entries:

$$\det \begin{pmatrix} a & * & * \\ 0 & b & * \\ 0 & 0 & c \end{pmatrix} = abc.$$

**Upper-triangular** means the only nonzero entries are on or above the diagonal.

Why?

- ▶ If one of the diagonal entries is zero, then the matrix has fewer than  $n$  pivots, so the RREF has a row of zeros. (Row operations don't change whether the determinant is zero.)
- ▶ Otherwise,

$$\begin{array}{ccc} \begin{pmatrix} a & * & * \\ 0 & b & * \\ 0 & 0 & c \end{pmatrix} & \xrightarrow{\text{scale by } a^{-1}, b^{-1}, c^{-1}} & \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} & \xrightarrow{\text{row replacements}} & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ \det = abc & & \det = 1 & & \det = 1 \end{array}$$

# Computing Determinants

## Method 1

### Theorem

Let  $A$  be a square matrix. Suppose you do some number of row operations on  $A$  to get a matrix  $B$  in row echelon form. Then

$$\det(A) = (-1)^r \frac{(\text{product of the diagonal entries of } B)}{(\text{product of the scaling factors})},$$

where  $r$  is the number of row swaps.

**Why?** Since  $B$  is in REF, it is upper-triangular, so its determinant is the product of its diagonal entries. You changed the determinant by  $(-1)^r$  and the product of the scaling factors when going from  $A$  to  $B$ .

### Remark

This is generally the fastest way to compute a determinant of a large matrix, either by hand or by computer.

Row reduction is  $O(n^3)$ ; cofactor expansion (next time) is  $O(n!) \sim O(n^n \sqrt{n})$ .

This is important in real life, when you're usually working with matrices with a gazillion columns.



# Computing Determinants

## Example

$$\begin{array}{l} \begin{pmatrix} 0 & -7 & -4 \\ 2 & 4 & 6 \\ 3 & 7 & -1 \end{pmatrix} \\ \begin{pmatrix} 2 & 4 & 6 \\ 0 & -7 & -4 \\ 3 & 7 & -1 \end{pmatrix} \\ \begin{pmatrix} 1 & 2 & 3 \\ 0 & -7 & -4 \\ 3 & 7 & -1 \end{pmatrix} \\ \begin{pmatrix} 1 & 2 & 3 \\ 0 & -7 & -4 \\ 0 & 1 & -10 \end{pmatrix} \\ \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & -10 \\ 0 & -7 & -4 \end{pmatrix} \\ \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & -10 \\ 0 & 0 & -74 \end{pmatrix} \end{array} \begin{array}{l} R_1 \leftrightarrow R_2 \\ \text{~~~~~} \\ R_1 = R_1 \div 2 \\ \text{~~~~~} \\ R_3 = R_3 - 3R_1 \\ \text{~~~~~} \\ R_2 \leftrightarrow R_3 \\ \text{~~~~~} \\ R_3 = R_3 + 7R_2 \\ \text{~~~~~} \end{array} \begin{array}{l} r = 1 \\ r = 1 \\ r = 1 \\ r = 1 \\ r = 2 \\ r = 2 \end{array} \begin{array}{l} \\ \text{scaling factors} = \frac{1}{2} \\ \text{scaling factors} = \frac{1}{2} \\ \text{scaling factors} = \frac{1}{2} \\ \text{scaling factors} = \frac{1}{2} \end{array}$$

$$\Rightarrow \det \begin{pmatrix} 0 & -7 & -4 \\ 1 & 4 & 6 \\ 3 & 7 & -1 \end{pmatrix} = (-1)^2 \frac{1 \cdot 1 \cdot -74}{1/2} = -148.$$

# Computing Determinants

## $2 \times 2$ Example

Let's compute the determinant of  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ , a general  $2 \times 2$  matrix.


- ▶ If  $a = 0$ , then

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \det \begin{pmatrix} 0 & b \\ c & d \end{pmatrix} = -\det \begin{pmatrix} c & d \\ 0 & b \end{pmatrix} = -bc.$$

- ▶ Otherwise,

$$\begin{aligned} \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} &= a \cdot \det \begin{pmatrix} 1 & b/a \\ c & d \end{pmatrix} = a \cdot \det \begin{pmatrix} 1 & b/a \\ 0 & d - c \cdot b/a \end{pmatrix} \\ &= a \cdot 1 \cdot (d - bc/a) = ad - bc. \end{aligned}$$

In both cases, the determinant magically turns out to be


$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

## Poll

Suppose that  $A$  is a  $4 \times 4$  matrix satisfying

$$Ae_1 = e_2 \quad Ae_2 = e_3 \quad Ae_3 = e_4 \quad Ae_4 = e_1.$$

What is  $\det(A)$ ?

- A.  $-1$       B.  $0$       C.  $1$

These equations tell us the columns of  $A$ :

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

You need 3 row swaps to transform this to the identity matrix.

So  $\det(A) = (-1)^3 = -1$ .

## Theorem

A square matrix  $A$  is invertible if and only if  $\det(A)$  is nonzero.

## Why?

- ▶ If  $A$  is invertible, then its reduced row echelon form is the identity matrix, which has determinant equal to 1.
- ▶ If  $A$  is not invertible, then its reduced row echelon form has a zero row, hence has zero determinant.
- ▶ Doing row operations doesn't change whether the determinant is zero.

# Determinants and Products

## Theorem

If  $A$  and  $B$  are two  $n \times n$  matrices, then

$$\det(AB) = \det(A) \cdot \det(B).$$

**Why?** If  $B$  is invertible, we can define

$$f(A) = \frac{\det(AB)}{\det(B)}.$$

Note  $f(I_n) = \det(I_n B) / \det(B) = 1$ . Check that  $f$  satisfies the same properties as  $\det$  with respect to row operations. So

$$\det(A) = f(A) = \frac{\det(AB)}{\det(B)} \implies \det(AB) = \det(A) \det(B).$$

What about if  $B$  is not invertible?

## Theorem

If  $A$  is invertible, then  $\det(A^{-1}) = \frac{1}{\det(A)}$ .

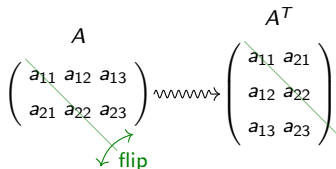
**Why?**  $I_n = AB \implies 1 = \det(I_n) = \det(AB) = \det(A) \det(B)$ .

# Transposes

## Review

**Recall:** The **transpose** of an  $m \times n$  matrix  $A$  is the  $n \times m$  matrix  $A^T$  whose rows are the columns of  $A$ . In other words, the  $ij$  entry of  $A^T$  is  $a_{ji}$ .

$$\begin{array}{ccc} A & & A^T \\ \left( \begin{array}{ccc} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{array} \right) & \rightsquigarrow & \left( \begin{array}{cc} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{array} \right) \end{array}$$



# Determinants and Transposes

## Theorem

If  $A$  is a square matrix, then

$$\det(A) = \det(A^T),$$

where  $A^T$  is the transpose of  $A$ .

**Example:**  $\det \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \det \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$ .

As a consequence,  $\det$  behaves the same way with respect to *column* operations as row operations.

**Corollary** ← an immediate consequence of a theorem

If  $A$  has a zero column, then  $\det(A) = 0$ .

## Corollary

The determinant of a *lower*-triangular matrix is the product of the diagonal entries.

(The transpose of a lower-triangular matrix is upper-triangular.)

# Section 5.3

Determinants and Volumes

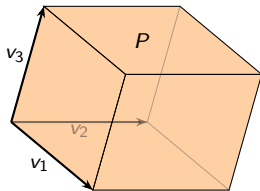
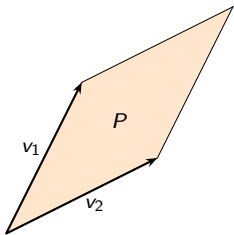


## Determinants and Volumes

Now we discuss a completely different description of (the absolute value of) the determinant, in terms of volumes.

This is a crucial component of the change-of-variables formula in multivariable calculus.

The columns  $v_1, v_2, \dots, v_n$  of an  $n \times n$  matrix  $A$  give you  $n$  vectors in  $\mathbf{R}^n$ . These determine a **parallelepiped**  $P$ .



### Theorem

Let  $A$  be an  $n \times n$  matrix with columns  $v_1, v_2, \dots, v_n$ , and let  $P$  be the parallelepiped determined by  $A$ . Then

$$(\text{volume of } P) = |\det(A)|.$$

## Theorem

Let  $A$  be an  $n \times n$  matrix with columns  $v_1, v_2, \dots, v_n$ , and let  $P$  be the parallelepiped determined by  $A$ . Then

$$(\text{volume of } P) = |\det(A)|.$$

**Sanity check:** the volume of  $P$  is zero  $\iff$  the columns are *linearly dependent* ( $P$  is “flat”)  $\iff$  the matrix  $A$  is not invertible.

**Why is the theorem true?** You only have to check that the volume behaves the same way under row operations as  $|\det|$  does.

Note that the volume of the unit cube (the parallelepiped defined by the identity matrix) is 1.

# Determinants and Volumes

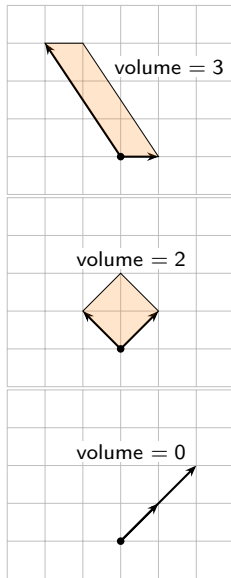
Examples in  $\mathbb{R}^2$

$$\det \begin{pmatrix} 1 & -2 \\ 0 & 3 \end{pmatrix} = 3$$

$$\det \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} = -2$$

(Should the volume really be  $-2$ ?)

$$\det \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} = 0$$



# Determinants and Volumes

## Theorem

Let  $A$  be an  $n \times n$  matrix with columns  $v_1, v_2, \dots, v_n$ , and let  $P$  be the parallelepiped determined by  $A$ . Then

$$(\text{volume of } P) = |\det(A)|.$$

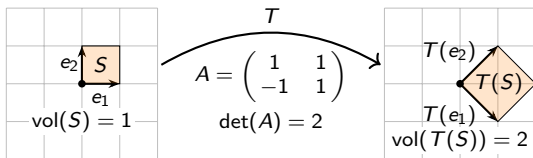
This is even true for curvy shapes, in the following sense.

## Theorem

Let  $A$  be an  $n \times n$  matrix, and let  $T(x) = Ax$ . If  $S$  is any region in  $\mathbf{R}^n$ , then

$$(\text{volume of } T(S)) = |\det(A)| (\text{volume of } S).$$

If  $S$  is the unit cube, then  $T(S)$  is the parallelepiped defined by the columns of  $A$ , since the columns of  $A$  are  $T(e_1), T(e_2), \dots, T(e_n)$ . In this case, the second theorem is the same as the first.



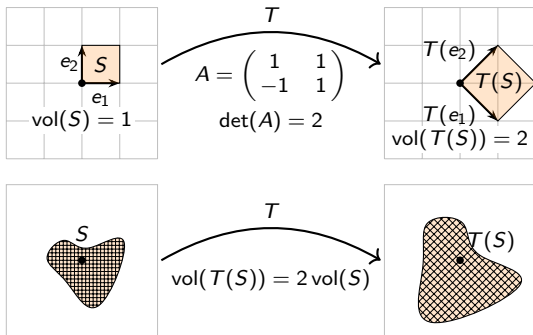
# Determinants and Volumes

## Theorem

Let  $A$  be an  $n \times n$  matrix, and let  $T(x) = Ax$ . If  $S$  is any region in  $\mathbf{R}^n$ , then

$$(\text{volume of } T(S)) = |\det(A)| (\text{volume of } S).$$

For curvy shapes, you break  $S$  up into a bunch of tiny cubes. Each one is scaled by  $|\det(A)|$ ; then you use *calculus* to reduce to the previous situation!



# Determinants and Volumes

## Example

### Theorem

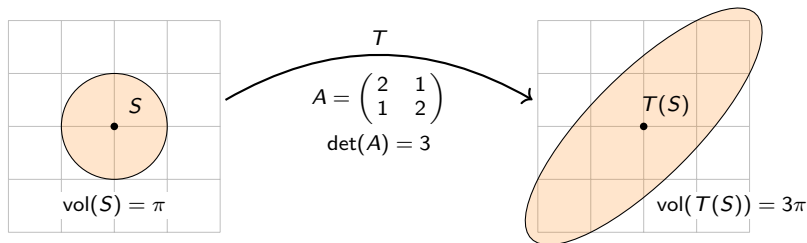
Let  $A$  be an  $n \times n$  matrix, and let  $T(x) = Ax$ . If  $S$  is any region in  $\mathbf{R}^n$ , then

$$(\text{volume of } T(S)) = |\det(A)| (\text{volume of } S).$$

**Example:** Let  $S$  be the unit disk in  $\mathbf{R}^2$ , and let  $T(x) = Ax$  for

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Note that  $\det(A) = 3$ .



## Magical Properties of the Determinant

you really have to know these

1. There is one and only one function  $\det: \{\text{square matrices}\} \rightarrow \mathbf{R}$  satisfying the properties (1)–(4) on the second slide.
2.  $A$  is invertible if and only if  $\det(A) \neq 0$ .
3. The determinant of an upper- or lower-triangular matrix is the product of the diagonal entries.
4. If we row reduce  $A$  to row echelon form  $B$  using  $r$  swaps, then

$$\det(A) = (-1)^r \frac{\text{(product of the diagonal entries of } B\text{)}}{\text{(product of the scaling factors)}}.$$

5.  $\det(AB) = \det(A)\det(B)$  and  $\det(A^{-1}) = \det(A)^{-1}$ .
6.  $\det(A) = \det(A^T)$ .
7.  $|\det(A)|$  is the volume of the parallelepiped defined by the columns of  $A$ .
8. If  $A$  is an  $n \times n$  matrix with transformation  $T(x) = Ax$ , and  $S$  is a subset of  $\mathbf{R}^n$ , then the volume of  $T(S)$  is  $|\det(A)|$  times the volume of  $S$ . (Even for curvy shapes  $S$ .)