Section 4.3

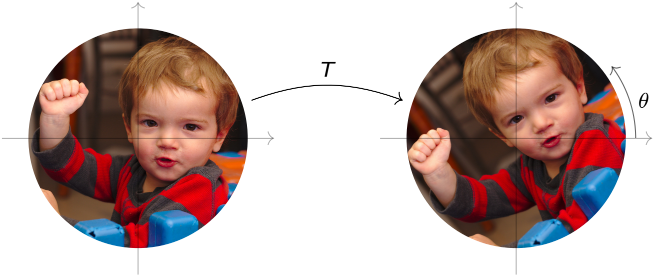
Linear Transformations

In the last two lectures we have been asking questions about transformations, and answering them in the case of matrix transformations.

However, sometimes it is not clear if a transformation is a matrix transformation or not.

Example

For a vector x in $\mathbf{R}^2$, let $T(x)$ be the counterclockwise rotation of x by an angle θ. Is $T(x) = Ax$ for some matrix A?



Today we will answer this question.

So, which transformations actually come from matrices?

Recall: If A is a matrix, u, v are vectors, and c is a scalar, then

$$A(u+v) = Au + Av \qquad A(cv) = cAv.$$

So if $T(x) = Ax$ is a matrix transformation then,

$$T(u+v) = T(u) + T(v) \quad \text{and} \quad T(cv) = cT(v).$$

Any matrix transformation has to satisfy this property. This property is so special that it has its own name.

Definition

A transformation $T\colon \mathbf{R}^n \to \mathbf{R}^m$ is **linear** if it satisfies the above equations for all vectors u, v in $\mathbf{R}^n$ and all scalars c.

In other words, T "respects" addition and scalar multiplication.

Check: if T is linear, then

$$T(0) = 0 \qquad T(cu + dv) = cT(u) + dT(v)$$

for all vectors u, v and scalars c, d. More generally,

$$T\big(c_1v_1 + c_2v_2 + \cdots + c_nv_n\big) = c_1T(v_1) + c_2T(v_2) + \cdots + c_nT(v_n).$$

In engineering this is called **superposition**.

Define $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ by $T(x) = 1.5x$. Is T linear? Check:

$$\begin{aligned} T(u+v) &= 1.5(u+v) = 1.5u + 1.5v = T(u) + T(v) \\ T(cv) &= 1.5(cv) = c(1.5v) = c(Tv). \end{aligned}$$

So T satisfies the two equations, hence T is linear.

Note: T is a matrix transformation!

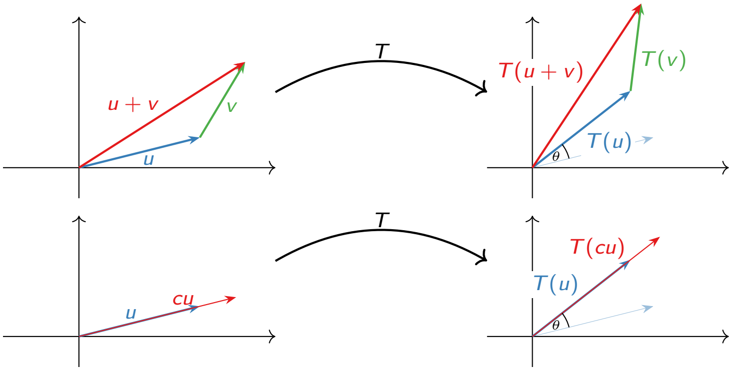
$$T(x) = \begin{pmatrix} 1.5 & 0 \\ 0 & 1.5 \end{pmatrix} x,$$

as we checked before.

Define $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ by

$$T(x) = \text{ the vector } x \text{ rotated counterclockwise by an angle of } \theta.$$

Is T linear? Check:



The pictures show $T(u) + T(v) = T(u+v)$ and $T(cu) = cT(u)$.

Since T satisfies the two equations, T is linear.

Is every transformation a linear transformation?

No! For instance, $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \sin x \\ xy \\ \cos y \end{pmatrix}$ is not linear.

Why? We have to check the two defining properties. Let's try the second:

$$T\left(c\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} \sin(cx) \\ (cx)(cy) \\ \cos(cy) \end{pmatrix} \stackrel{?}{=} c\begin{pmatrix} \sin x \\ xy \\ \cos y \end{pmatrix} = cT\begin{pmatrix} x \\ y \end{pmatrix}$$

Not necessarily: if $c = 2$ and $x = \pi$, $y = \pi$, then

$$T\left(2\begin{pmatrix} \pi \\ \pi \end{pmatrix}\right) = T\begin{pmatrix} 2\pi \\ 2\pi \end{pmatrix} = \begin{pmatrix} \sin 2\pi \\ 2\pi \cdot 2\pi \\ \cos 2\pi \end{pmatrix} = \begin{pmatrix} 0 \\ 4\pi^2 \\ 1 \end{pmatrix}$$

$$2T\begin{pmatrix} \pi \\ \pi \end{pmatrix} = 2\begin{pmatrix} \sin \pi \\ \pi \cdot \pi \\ \cos \pi \end{pmatrix} = \begin{pmatrix} 0 \\ 2\pi^2 \\ -2 \end{pmatrix}.$$

So T fails the second property. Conclusion: T is *not* a matrix transformation!

(We could also have noted $T(0) \neq 0$.)

Poll

Which of the following transformations are linear?

$$\textbf{A. } T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} |x_1| \\ x_2 \end{pmatrix} \qquad \textbf{B. } T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2x_1 + x_2 \\ x_1 - 2x_2 \end{pmatrix}$$

$$\textbf{C. } T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 x_2 \\ x_2 \end{pmatrix} \qquad \textbf{D. } T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2x_1 + 1 \\ x_1 - 2x_2 \end{pmatrix}$$

A. $T\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \neq \begin{pmatrix} 2 \\ 0 \end{pmatrix} = T\begin{pmatrix} 1 \\ 0 \end{pmatrix} + T\begin{pmatrix} -1 \\ 0 \end{pmatrix}$, so not linear.

B. Linear.

C. $T\left(2\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 4 \\ 2 \end{pmatrix} \neq 2T\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, so not linear.

D. $T\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \neq 0$, so not linear.

Remark: in fact, T is linear if and only if each entry of the output is a linear function of the entries of the input, with no constant terms. Check this!

The Matrix of a Linear Transformation

We will see that a *linear* transformation T is a matrix transformation: $T(x) = Ax$.

But what matrix does T come from? What is A?

Here's how to compute it.

Unit Coordinate Vectors

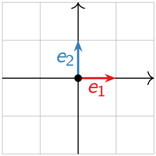
Definition

The **unit coordinate vectors** in $\mathbf{R}^n$ are

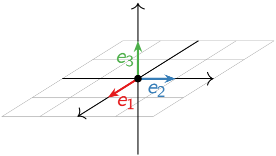
This is what $e_1, e_2, \ldots$ mean, for the rest of the class.

$$\boxed{e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad \ldots, \quad e_{n-1} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \end{pmatrix}, \quad e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.}$$

in $\mathbf{R}^2$



in $\mathbf{R}^3$



Note: if A is an $m \times n$ matrix with columns $v_1, v_2, \ldots, v_n$, then $Ae_i = v_i$ for $i = 1, 2, \ldots, n$: multiplying a matrix by e_i gives you the ith column.

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix} \qquad \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix} \qquad \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix}$$

Linear Transformations are Matrix Transformations

Recall: A matrix A defines a linear transformation T by $T(x) = Ax$.

Theorem

Let $T\colon \mathbf{R}^n \to \mathbf{R}^m$ be a linear transformation. Let

$$A = \begin{pmatrix} | & | & & | \\ T(e_1) & T(e_2) & \cdots & T(e_n) \\ | & | & & | \end{pmatrix}.$$

This is an $m \times n$ matrix, and T is the matrix transformation for A: $T(x) = Ax$.

The matrix A is called the **standard matrix** for T.

Take-Away

Linear transformations are the same as matrix transformations.

Dictionary

Linear transformation $T\colon \mathbf{R}^n \to \mathbf{R}^m$	$\rightsquigarrow$	$m \times n$ matrix $A = \begin{pmatrix} \vert & \vert & & \vert \\ T(e_1) & T(e_2) & \cdots & T(e_n) \\ \vert & \vert & & \vert \end{pmatrix}$
$T(x) = Ax$ $T\colon \mathbf{R}^n \to \mathbf{R}^m$	$\leftsquigarrow$	$m \times n$ matrix A

Why is a linear transformation a matrix transformation?

Suppose for simplicity that $T\colon \mathbf{R}^3 \to \mathbf{R}^2$.

$$\begin{aligned}
T\begin{pmatrix} x \\ y \\ z \end{pmatrix} &= T\left(x\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right) \\
&= T\big(xe_1 + ye_2 + ze_3\big) \\
&= xT(e_1) + yT(e_2) + zT(e_3) \\
&= \begin{pmatrix} | & | & | \\ T(e_1) & T(e_2) & T(e_3) \\ | & | & | \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\
&= A\begin{pmatrix} x \\ y \\ z \end{pmatrix}.
\end{aligned}$$

Before, we defined a **dilation** transformation $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ by $T(x) = 1.5x$. What is its standard matrix?

$$\left.\begin{aligned} T(e_1) &= 1.5e_1 = \begin{pmatrix} 1.5 \\ 0 \end{pmatrix} \\ T(e_2) &= 1.5e_2 = \begin{pmatrix} 0 \\ 1.5 \end{pmatrix} \end{aligned}\right\} \implies A = \begin{pmatrix} 1.5 & 0 \\ 0 & 1.5 \end{pmatrix}.$$

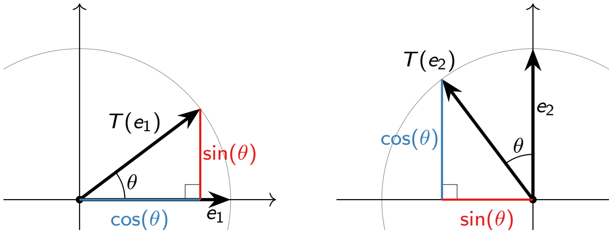
Check:

$$\begin{pmatrix} 1.5 & 0 \\ 0 & 1.5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1.5x \\ 1.5y \end{pmatrix} = 1.5\begin{pmatrix} x \\ y \end{pmatrix} = T\begin{pmatrix} x \\ y \end{pmatrix}.$$

Question

What is the matrix for the linear transformation $T \colon \mathbf{R}^2 \to \mathbf{R}^2$ defined by

$$T(x) = x \text{ rotated counterclockwise by an angle } \theta?$$



$$\left.\begin{array}{l} T(e_1) = \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} \\[1em] T(e_2) = \begin{pmatrix} -\sin(\theta) \\ \cos(\theta) \end{pmatrix} \end{array}\right\} \implies A = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \qquad \left(\begin{array}{c} \theta = 90^\circ \implies \\ A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\ \text{from before} \end{array} \right)$$

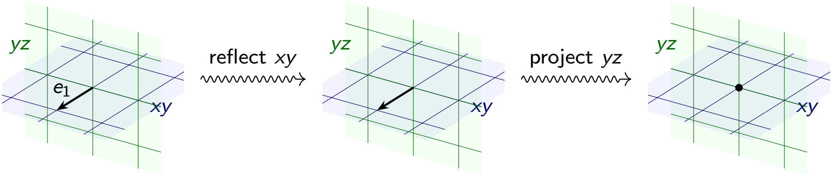
Linear Transformations are Matrix Transformations

Example

Question

What is the matrix for the linear transformation $T\colon \mathbf{R}^3 \to \mathbf{R}^3$ that reflects through the xy-plane and then projects onto the yz-plane?

[interactive]

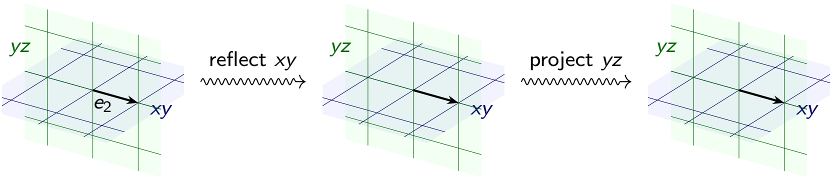


$$T(e_1) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Question

What is the matrix for the linear transformation $T\colon \mathbf{R}^3 \to \mathbf{R}^3$ that reflects through the xy-plane and then projects onto the yz-plane?

[interactive]

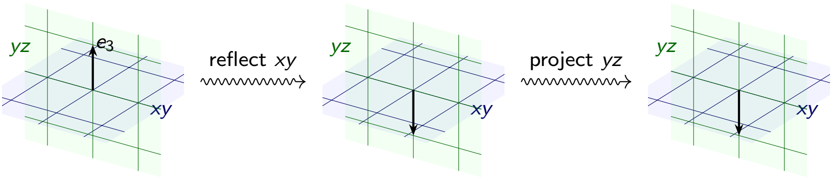


$$T(e_2) = e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

Question

What is the matrix for the linear transformation $T\colon \mathbf{R}^3 \to \mathbf{R}^3$ that reflects through the xy-plane and then projects onto the yz-plane?

[interactive]



$$T(e_3) = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}.$$

Linear Transformations are Matrix Transformations

Example, continued

Question

What is the matrix for the linear transformation $T\colon \mathbf{R}^3 \to \mathbf{R}^3$ that reflects through the xy-plane and then projects onto the yz-plane?

$$\left.\begin{array}{l} T(e_1) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\[2ex] T(e_2) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \\[2ex] T(e_1) = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \end{array}\right\} \implies A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Question

Define a linear transformation $T\colon \mathbf{R}^3 \to \mathbf{R}^2$ by

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + 2y + 3z \\ -y - 5z \end{pmatrix}.$$

What is the standard matrix A for T?

$$T(e_1) = T\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad T(e_2) = T\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \qquad T(e_3) = T\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ -5 \end{pmatrix}$$

$$\implies A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -5 \end{pmatrix}.$$

Questions About Linear Transformations

A linear transformation is a matrix transformation, so questions about linear transformations are questions about matrices.

Question

Let $T\colon \mathbf{R}^3 \to \mathbf{R}^3$ be the linear transformation that reflects through the xy-plane and then projects onto the yz-plane. Is T one-to-one?

We have $T(x) = Ax$ for

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

This does not have a pivot in the first column, so T is not one-to-one.

Summary

- **Linear transformations** are the transformations that come from matrices.
- The **unit coordinate vectors** $e_1, e_2, \ldots$ are the unit vectors in the positive direction along the coordinate axes.
- You compute the columns of the matrix for a linear transformation by plugging in the unit coordinate vectors.
- This is useful when the transformation is specified geometrically, in terms of a formula, or any other way that isn't as a matrix transformation.
- There are lots of equivalent conditions for a linear transformation to be one-to-one and/or onto, in terms of its matrix.