

Sections 3.7 and 3.9

Basis, Dimension, and the Rank and Basis Theorems

Subspaces

Reminder

Recall: a subspace of \mathbf{R}^n is the same thing as a span, except we haven't computed a spanning set yet.

For example, $\text{Col } A$ and $\text{Nul } A$ for a matrix A .

There are lots of choices of spanning set for a given subspace.

Are some better than others?

What is the *smallest number* of vectors that are needed to span a subspace?

Definition

Let V be a subspace of \mathbf{R}^n . A **basis** of V is a set of vectors $\{v_1, v_2, \dots, v_m\}$ in V such that:

1. $V = \text{Span}\{v_1, v_2, \dots, v_m\}$, and
2. $\{v_1, v_2, \dots, v_m\}$ is linearly independent.

The number of vectors in a basis is the **dimension** of V , and is written $\dim V$.

Note the big
red border here

Why is a basis the smallest number of vectors needed to span?

Recall: *linearly independent* means that every time you add another vector, the span gets bigger.

Hence, if we remove any vector, the span gets *smaller*: so any smaller set can't span V .

Important

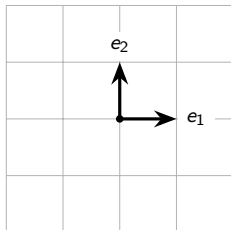
A subspace has *many different* bases, but they all have the same number of vectors.

Question

What is a basis for \mathbf{R}^2 ?

We need two vectors that *span* \mathbf{R}^2 and are *linearly independent*. $\{e_1, e_2\}$ is one basis.

1. They span: $\begin{pmatrix} a \\ b \end{pmatrix} = ae_1 + be_2$.
2. They are linearly independent because they are not collinear.

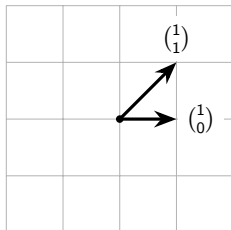


Question

What is another basis for \mathbf{R}^2 ?

Any two nonzero vectors that are not collinear. $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$ is also a basis.

1. They span: $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ has a pivot in every row.
2. They are linearly independent: $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ has a pivot in every column.



The unit coordinate vectors

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad \dots, \quad e_{n-1} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \end{pmatrix}, \quad e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

are a basis for \mathbf{R}^n . The identity matrix has columns e_1, e_2, \dots, e_n .

1. They span: I_n has a pivot in every row.
2. They are linearly independent: I_n has a pivot in every column.

In general: $\{v_1, v_2, \dots, v_n\}$ is a basis for \mathbf{R}^n if and only if the matrix

$$A = \begin{pmatrix} | & | & \cdots & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & \cdots & | \end{pmatrix}$$

has a pivot in every row and every column, i.e. if A is *invertible*.

Sanity check: we have shown that $\dim \mathbf{R}^n = n$.

Basis of a Subspace

Example

Example

Let

$$V = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ in } \mathbf{R}^3 \mid x + 3y + z = 0 \right\} \quad \mathcal{B} = \left\{ \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix} \right\}.$$

Verify that \mathcal{B} is a basis for V . (So $\dim V = 2$: it is a *plane*.) [\[interactive\]](#)

0. **In V :** both vectors are in V because

$$-3 + 3(1) + 0 = 0 \quad \text{and} \quad 0 + 3(1) + (-3) = 0.$$

1. **Span:** If $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is in V , then $y = -\frac{1}{3}(x + z)$, so

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -\frac{x}{3} \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} - \frac{z}{3} \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix}.$$

2. **Linearly independent:**

$$c_1 \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix} = \mathbf{0} \implies \begin{pmatrix} -3c_1 \\ c_1 + c_2 \\ -3c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \implies c_1 = c_2 = 0.$$

Fact

The vectors in the parametric vector form of the general solution to $Ax = 0$ always form a basis for Nul A .

Example

$$A = \begin{pmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{\text{parametric vector form}} x = x_3 \begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix} \xrightarrow{\text{basis of Nul } A} \left\{ \begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix} \right\}$$

1. The vectors span Nul A by construction (every solution to $Ax = 0$ has this form).
2. Can you see why they are linearly independent? (Look at the last two rows.)

Fact

The *pivot columns* of A always form a basis for Col A .

Warning: I mean the pivot columns of the *original* matrix A , not the row-reduced form. (Row reduction changes the column space.)

Example

$$A = \begin{pmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

So a basis for Col A is

$$\left\{ \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} \right\}.$$

Why? See slides on linear independence.

The Rank Theorem

Recall:

- ▶ The **dimension** of a subspace V is the number of vectors in a basis for V .
- ▶ A basis for the column space of a matrix A is given by the pivot columns.
- ▶ A basis for the null space of A is given by the vectors attached to the free variables in the parametric vector form.

Definition

The **rank** of a matrix A , written $\text{rank } A$, is the dimension of the column space $\text{Col } A$. The **nullity** of A , written $\text{nullity } A$, is the dimension of the solution set of $Ax = 0$.

Observe:

$\text{rank } A = \dim \text{Col } A =$ the number of columns with pivots

$\text{nullity } A = \dim \text{Nul } A =$ the number of free variables

$=$ the number of columns without pivots.

Rank Theorem

If A is an $m \times n$ matrix, then

$$\text{rank } A + \text{nullity } A = n = \text{the number of columns of } A.$$

In other words, [\[interactive 1\]](#) [\[interactive 2\]](#)

(dimension of column span) + (dimension of solution set) = (number of variables).

The Rank Theorem

Example

$$A = \begin{pmatrix} \boxed{1} & \boxed{2} & 0 & -1 \\ \boxed{-2} & \boxed{-3} & 4 & 5 \\ \boxed{2} & \boxed{4} & 0 & -2 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & \boxed{-8} & \boxed{-7} \\ 0 & 1 & \boxed{4} & \boxed{3} \\ 0 & 0 & \boxed{0} & \boxed{0} \end{pmatrix}$$

basis of Col A free variables

A basis for Col A is

$$\left\{ \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} \right\},$$

so $\text{rank } A = \dim \text{Col } A = 2$.

Since there are two free variables x_3, x_4 , the parametric vector form for the solutions to $Ax = 0$ is

$$x = x_3 \begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix} \xrightarrow{\text{basis for Nul } A} \left\{ \begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Thus $\text{nullity } A = \dim \text{Nul } A = 2$.

The Rank Theorem says $2 + 2 = 4$.

The Basis Theorem

Basis Theorem

Let V be a subspace of dimension m . Then:

- ▶ Any m linearly independent vectors in V form a basis for V .
- ▶ Any m vectors that span V form a basis for V .

Upshot

If you *already* know that $\dim V = m$, and you have m vectors $\mathcal{B} = \{v_1, v_2, \dots, v_m\}$ in V , then you only have to check *one* of

1. \mathcal{B} is linearly independent, *or*
2. \mathcal{B} spans V

in order for \mathcal{B} to be a basis.

Example: any three linearly independent vectors form a basis for \mathbf{R}^3 .

Summary

- ▶ A **basis** of a subspace is a minimal set of spanning vectors.
- ▶ There are recipes for computing a basis for the column space and null space of a matrix.
- ▶ The **dimension** of a subspace is the number of vectors in any basis.
- ▶ The **rank theorem** says the dimension of the column space of a matrix, plus the dimension of the null space, is the number of columns of the matrix.
- ▶ The **basis theorem** says that if you already know that $\dim V = m$, and you have m vectors in V , then you only have to check if they span *or* they're linearly independent to know they're a basis.