

# Section 3.6

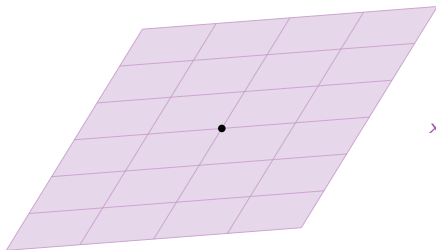
## Subspaces

# Motivation

Today we will discuss **subspaces** of  $\mathbf{R}^n$ .

A subspace turns out to be the same as a span, except we don't know *which* vectors it's the span of.

This arises naturally when you have, say, a plane through the origin in  $\mathbf{R}^3$  which is *not* defined (a priori) as a span, but you still want to say something about it.



$$x + 3y + z = 0$$

# Definition of Subspace

## Definition

A **subspace** of  $\mathbf{R}^n$  is a subset  $V$  of  $\mathbf{R}^n$  satisfying:

1. The zero vector is in  $V$ . "not empty"
2. If  $u$  and  $v$  are in  $V$ , then  $u + v$  is also in  $V$ . "closed under addition"
3. If  $u$  is in  $V$  and  $c$  is in  $\mathbf{R}$ , then  $cu$  is in  $V$ . "closed under  $\times$  scalars"

Fast-forward

Every subspace is a span, and every span is a subspace.

A subspace is a span of some vectors, but you haven't computed what those vectors are yet.

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## What does this mean?

- ▶ If  $v$  is in  $V$ , then all scalar multiples of  $v$  are in  $V$  by (3). That is, the line through  $v$  is in  $V$ .
- ▶ If  $u, v$  are in  $V$ , then  $xu$  and  $yv$  are in  $V$  for scalars  $x, y$  by (3). So  $xu + yv$  is in  $V$  by (2). So  $\text{Span}\{u, v\}$  is contained in  $V$ .
- ▶ Likewise, if  $v_1, v_2, \dots, v_n$  are all in  $V$ , then  $\text{Span}\{v_1, v_2, \dots, v_n\}$  is contained in  $V$ : a subspace contains the span of any set of vectors in it.

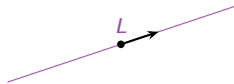
If you pick enough vectors in  $V$ , eventually their span will fill up  $V$ , so:

A subspace is a span of some set of vectors in it.

# Examples

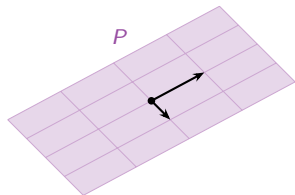
## Example

A line  $L$  through the origin: this contains the span of any vector in  $L$ .



## Example

A plane  $P$  through the origin: this contains the span of any vectors in  $P$ .



## Example

All of  $\mathbf{R}^n$ : this contains  $0$ , and is closed under addition and scalar multiplication.

## Example

The subset  $\{0\}$ : this subspace contains only one vector.

Note these are all pictures of spans! (Line, plane, space, etc.)

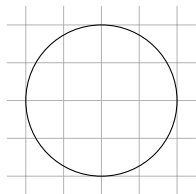
# Subsets and Subspaces

They aren't the same thing

A **subset** of  $\mathbf{R}^n$  is any collection of vectors whatsoever.

All of the following non-examples are still subsets.

A **subspace** is a special kind of subset, which satisfies the three defining properties.



Subset: *yes*

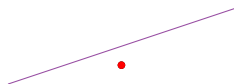
Subspace: *no*

# Non-Examples

## Non-Example

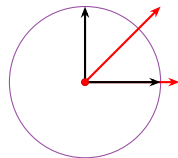
A line  $L$  (or any other set) that doesn't contain the origin is not a subspace.

Fails: 1.



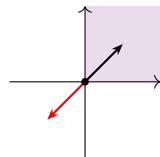
## Non-Example

A circle  $C$  is not a subspace. Fails: 1,2,3. Think: a circle isn't a "linear space."



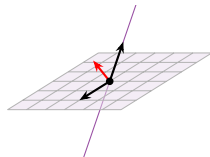
## Non-Example

The first quadrant in  $\mathbf{R}^2$  is not a subspace. Fails: 3 only.



## Non-Example

A line union a plane in  $\mathbf{R}^3$  is not a subspace. Fails: 2 only.



# Spans are Subspaces

## Theorem

Any Span $\{v_1, v_2, \dots, v_n\}$  is a subspace.

!!!

Every subspace is a span, and every span is a subspace.

## Definition

If  $V = \text{Span}\{v_1, v_2, \dots, v_n\}$ , we say that  $V$  is the subspace **generated by** or **spanned by** the vectors  $v_1, v_2, \dots, v_n$ .

## Check:

1.  $0 = 0v_1 + 0v_2 + \dots + 0v_n$  is in the span.
2. If, say,  $u = 3v_1 + 4v_2$  and  $v = -v_1 - 2v_2$ , then

$$u + v = 3v_1 + 4v_2 - v_1 - 2v_2 = 2v_1 + 2v_2$$

is also in the span.

3. Similarly, if  $u$  is in the span, then so is  $cu$  for any scalar  $c$ .



## Poll

Which of the following are subspaces?

- A. The empty set  $\{\}$ .
- B. The solution set to a homogeneous system of linear equations.
- C. The solution set to an inhomogeneous system of linear equations.
- D. The set of all vectors in  $\mathbf{R}^n$  with rational (fraction) coordinates.

For the ones which are not subspaces, which property(ies) do they not satisfy?

- A. This is not a subspace: it does not contain the zero vector.
- B. This is a subspace: the solution set is a span, produced by finding the parametric vector form of the solution.
- C. This is not a subspace: it does not contain 0.
- D. This is not a subspace: it is not closed under multiplication by scalars (e.g. by  $\pi$ ).

# Subspaces

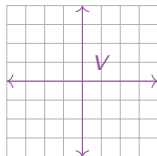
## Verification

Let  $V = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \text{ in } \mathbf{R}^2 \mid ab = 0 \right\}$ . Let's check if  $V$  is a subspace or not.

1. Does  $V$  contain the zero vector?  $\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies ab = 0$  ✓

3. Is  $V$  closed under scalar multiplication?

- ▶ Let  $\begin{pmatrix} a \\ b \end{pmatrix}$  be (an unknown vector) in  $V$ .
- ▶ *This means:*  $a$  and  $b$  are numbers such that  $ab = 0$ .
- ▶ Let  $c$  be a scalar. Is  $c\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} ca \\ cb \end{pmatrix}$  in  $V$ ?
- ▶ *This means:*  $(ca)(cb) = 0$ .
- ▶ Well,  $(ca)(cb) = c^2(ab) = c^2(0) = 0$  ✓



2. Is  $V$  closed under addition?

- ▶ Let  $\begin{pmatrix} a \\ b \end{pmatrix}$  and  $\begin{pmatrix} a' \\ b' \end{pmatrix}$  be (unknown vectors) in  $V$ .
- ▶ *This means:*  $ab = 0$ , and  $a'b' = 0$ .
- ▶ Is  $\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} a' \\ b' \end{pmatrix} = \begin{pmatrix} a+a' \\ b+b' \end{pmatrix}$  in  $V$ ?
- ▶ *This means:*  $(a+a')(b+b') = 0$ .
- ▶ This is not true for all such  $a, a', b, b'$ : for instance,  $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$  and  $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$  are in  $V$ , but their sum  $\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$  is not in  $V$ , because  $1 \cdot 1 \neq 0$ . ✗

We conclude that  $V$  is *not* a subspace. A picture is above. (It doesn't look like a span.)

## Column Space and Null Space

An  $m \times n$  matrix  $A$  naturally gives rise to *two* subspaces.

### Definition

- ▶ The **column space** of  $A$  is the subspace of  $\mathbf{R}^m$  spanned by the columns of  $A$ . It is written  $\text{Col } A$ .
- ▶ The **null space** of  $A$  is the set of all solutions of the homogeneous equation  $Ax = 0$ :

$$\text{Nul } A = \{x \text{ in } \mathbf{R}^n \mid Ax = 0\}.$$

This is a subspace of  $\mathbf{R}^n$ .

The column space is defined as a span, so we know it is a subspace. It is the range (as opposed to the codomain) of the transformation  $T(x) = Ax$ .

**Check** that the null space is a subspace:

1.  $0$  is in  $\text{Nul } A$  because  $A0 = 0$ .
2. If  $u$  and  $v$  are in  $\text{Nul } A$ , then  $Au = 0$  and  $Av = 0$ . Hence

$$A(u + v) = Au + Av = 0,$$

so  $u + v$  is in  $\text{Nul } A$ .

3. If  $u$  is in  $\text{Nul } A$ , then  $Au = 0$ . For any scalar  $c$ ,  $A(cu) = cAu = 0$ . So  $cu$  is in  $\text{Nul } A$ .

# Column Space and Null Space

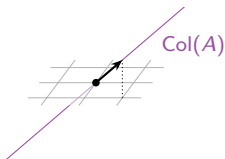
## Example

$$\text{Let } A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Let's compute the column space:

$$\text{Col } A = \text{Span} \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\} = \text{Span} \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

This is a line in  $\mathbf{R}^3$ .



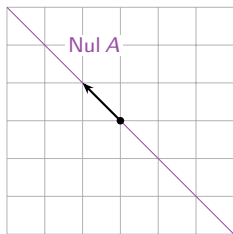
Let's compute the null space:

The reduced row echelon form of  $A$  is  $\begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$ .

This gives the equation  $x + y = 0$ , or

$$\begin{array}{l} x = -y \\ y = y \end{array} \quad \begin{array}{l} \text{parametric vector form} \\ \text{~~~~~} \end{array} \quad \begin{pmatrix} x \\ y \end{pmatrix} = y \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Hence the null space is  $\text{Span}\left\{\begin{pmatrix} -1 \\ 1 \end{pmatrix}\right\}$ , a line in  $\mathbf{R}^2$ .



## The Null Space is a Span

The column space of a matrix  $A$  is defined to be a span (of the columns).

The null space is defined to be the solution set to  $Ax = 0$ . It is a subspace, so it is a span.

### Question

How to find vectors which span the null space?

**Answer:** Parametric vector form! We know that the solution set to  $Ax = 0$  has a parametric form that looks like

$$x_3 \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -2 \\ 3 \\ 0 \\ 1 \end{pmatrix} \quad \text{if, say, } x_3 \text{ and } x_4 \text{ are the free variables. So} \quad \text{Nul } A = \text{Span} \left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 3 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Refer back to the slides for §3.4 (Solution Sets).

**Note:** It is much easier to define the null space first as a subspace, then find spanning vectors *later*, if we need them. This is one reason subspaces are so useful.

# Subspaces

## Summary

- ▶ A **subspace** is the same as a span of some number of vectors, but we haven't computed the vectors yet.
- ▶ To any matrix is associated two subspaces, the **column space** and the **null space**:

$\text{Col } A =$  the span of the columns of  $A$

$\text{Nul } A =$  the solution set of  $Ax = 0$ .

How do you check if a subset is a subspace?

- ▶ Is it a span? Can it be written as a span?
- ▶ Can it be written as the column space of a matrix?
- ▶ Can it be written as the null space of a matrix?
- ▶ Is it all of  $\mathbf{R}^n$  or the zero subspace  $\{0\}$ ?
- ▶ Can it be written as a type of subspace that we'll learn about later (eigenspaces, ...)?

If so, then it's automatically a subspace.

If all else fails:

- ▶ Can you verify directly that it satisfies the three defining properties?