Math 1553 Worksheet: 4.1-4.3

1. Answer true if the statement is *always* true. Otherwise, answer false. Justify your answer.

Suppose v_1, v_2, and v_3 are vectors in $\mathbf{R}^3$ and the volume of the parallelepiped naturally formed by v_1, v_2, and v_3 is 10. Then Span$\{v_1, v_2, v_3\}$ is all of $\mathbf{R}^3$.

Solution.

True. Let $A = \begin{pmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{pmatrix}$. We know $\det(A) = 10 \neq 0$, so A is invertible. Therefore, every vector in $\mathbf{R}^3$ can be written as a linear combination of v_1, v_2, v_3.

2. Find the volume of the parallelepiped naturally formed by $\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$, and $\begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$ using a cofactor expansion.

Solution.

We expand along the first row:

$$\det\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 3 \\ -2 & 1 & 1 \end{pmatrix} = 2\det\begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix} - 1\det\begin{pmatrix} 1 & 3 \\ -2 & 1 \end{pmatrix} + 1\det\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$$
$$= 2(2-3) - 1(1+6) + 1(1+4)$$
$$= -2 - 7 + 5 = -4.$$

The volume is $|-4| = 4$.

3. Let $A = \begin{pmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{pmatrix}$.

a) Compute $\det(A)$ using row reduction.

b) Compute $\det(A^{-1})$ without doing any more work.

c) Compute $\det((A^T)^5)$ without doing any more work.

Solution.

a) Below, r counts the row swaps and s measures the scaling factors.

$$\begin{pmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{pmatrix} \xrightarrow{R_1 = \frac{R_1}{2}} \begin{pmatrix} 1 & -4 & 3 & 4 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{pmatrix} \quad (r = 0,\ s = \frac{1}{2})$$

$$\xrightarrow[R_3 = R_3 + 3R_1,\ R_4 = R_4 - R_1]{R_2 = R_2 - 3R_1} \begin{pmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & -12 & 10 & 10 \\ 0 & 0 & -3 & 2 \end{pmatrix} \quad (r = 0,\ s = \frac{1}{2})$$

$$\xrightarrow{R_3 = R_3 + 4R_2} \begin{pmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & 0 & -6 & 2 \\ 0 & 0 & -3 & 2 \end{pmatrix} \quad (r = 0,\ s = \frac{1}{2})$$

$$\xrightarrow{R_4 = R_4 - \frac{R_2}{2}} \begin{pmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & 0 & -6 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (r = 0,\ s = \frac{1}{2})$$

$$\det(A) = (-1)^0 \frac{1 \cdot 3 \cdot (-6) \cdot 1}{1/2} = -36.$$

b) From our notes, we know $\det(A^{-1}) = \dfrac{1}{\det(A)} = -\dfrac{1}{36}$.

c) $\det(A^T) = \det(A) = -36$. By the multiplicative property of determinants, if B is any $n \times n$ matrix, then $\det(B^n) = (\det B)^n$, so

$$\det((A^T)^5) = (\det A^T)^5 = (-36)^5 = -60,466,176$$

4. Play matrix tic-tac-toe!

Instead of X against O, we have 1 against 0. The 1-player wins if the final matrix has nonzero determinant, while the 0-player wins if the determinant is zero. You can change who goes first, and you can also modify the size of the matrix.

Click the link above, or copy and paste the url below:

`http://textbooks.math.gatech.edu/ila/demos/tictactoe/tictactoe.html`

Can you think of a winning strategy for the 0 player who goes first in the 2×2 case?
Is there a winning strategy for the 1 player if they go first in the 2×2 case?