Math 1553 Worksheet §§5.6-6.3

1. Courage Soda and Dexter Soda compete for a market of 210 customers who drink soda each day. Today, Courage has 80 customers and Dexter has 130 customers. Each day:

70% of Courage Soda's customers keep drinking Courage Soda, while 30% switch to Dexter Soda.

40% of Dexter Soda's customers keep drinking Dexter Soda, while 60% switch to Courage Soda.

a) Write a stochastic matrix A and a vector x so that Ax will give the number of customers for Courage Soda and Dexter Soda (in that order) tomorrow.
You do not need to compute Ax.

$$A = \begin{pmatrix} 0.7 & 0.6 \\ 0.3 & 0.4 \end{pmatrix} \text{ and } x = \begin{pmatrix} 80 \\ 130 \end{pmatrix}.$$

b) A quick computation shows that the 1-eigenspace for this positive stochastic matrix A is spanned by $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

Find the steady-state vector for A. In the long run, roughly how many daily customers will Courage Soda have?

The steady state vector is $w = \dfrac{1}{2+1}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix}$.

As n gets large, $A^n \begin{pmatrix} 80 \\ 130 \end{pmatrix}$ approaches $210\begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix} = \begin{pmatrix} 140 \\ 70 \end{pmatrix}$. Courage will have roughly 140 customers.

2. Let W be the set of all vectors in $\mathbf{R}^3$ of the form $(x, x-y, y)$ where x and y are real numbers.

a) Find a basis for $W^\perp$.

b) Find the matrix B for orthogonal projection onto W.

Solution.

a) A vector in W has the form

$$\begin{pmatrix} x \\ x-y \\ y \end{pmatrix} = \begin{pmatrix} x \\ x \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -y \\ y \end{pmatrix} = x\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + y\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \quad \text{so } W \text{ has basis } \left\{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}\right\}.$$

To get $W^\perp$ we find $\operatorname{Nul}\begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$ which gives us

$$x_1 = -x_3, \quad x_2 = x_3, \quad x_3 = x_3 \text{ (free)},$$

so $W^\perp$ has basis $\left\{\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}\right\}$.

b) Let A be the matrix whose columns are the basis vectors for W: $A = \begin{pmatrix} 1 & 0 \\ 1 & -1 \\ 0 & 1 \end{pmatrix}$.

We calculate $A^TA = \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$, so

$$B = A(A^TA)^{-1}A^T = \begin{pmatrix} 1 & 0 \\ 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}^{-1}\begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 1 & -1 \\ 0 & 1 \end{pmatrix}\frac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}.$$