MATH 1553, JANKOWSKI
MIDTERM 1, FALL 2019

Name		**Section**	

Please **read all instructions** carefully before beginning.

- Write your name on the front of each page (not just the cover page!).
- The maximum score on this exam is 50 points, and you have 50 minutes to complete this exam.
- There are no calculators or aids of any kind (notes, text, etc.) allowed.
- As always, RREF means “reduced row echelon form.”
- Show your work, unless instructed otherwise. A correct answer without appropriate work will receive little or no credit! If you cannot fit your work on the front side of the page, use the back side of the page and indicate that you are using the back side.
- We will hand out loose scrap paper, but it **will not be graded** under any circumstances. All work must be written on the exam itself.
- You may cite any theorem proved in class or in the sections we covered the text.
- Good luck!

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Problem 1.

[10 points]

These problems are true or false. Circle **T** if the statement is *always* true. Otherwise, circle **F**. You do not need to justify your answer.

a) **T** **F** The augmented matrix below is in RREF.

$$\left(\begin{array}{ccc|c} 1 & 3 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

b) **T** **F** If the RREF of an augmented matrix has a row of zeros, then the corresponding linear system of equations either has no solutions or infinitely many solutions.

c) **T** **F** Suppose $v_1,\ v_2,\ v_3,\ b$ are vectors in $\mathbf{R}^3$. If the vector equation

$$x_1v_1 + x_2v_2 + x_3v_3 = b$$

is inconsistent, then the vector equation $x_1v_1 + x_2v_2 = b$ must also be inconsistent.

d) **T** **F** If A is a 2×3 matrix and the solution set to $Ax = 0$ is a plane in $\mathbf{R}^3$, then the equation $Ax = b$ must be inconsistent for some b in $\mathbf{R}^2$.

e) **T** **F** If A is a 2×2 matrix and the equation $Ax = \begin{pmatrix} -2 \\ -1 \end{pmatrix}$ is consistent, then the vector $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ must be in the span of the columns of A.

Solution.

a) True.

b) False. For example, the augmented system $\left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right)$ has a unique solution.

c) True. If b is not in $\mathrm{Span}\{v_1, v_2, v_3\}$ then it cannot be in $\mathrm{Span}\{v_1, v_2\}$, every vector in $\mathrm{Span}\{v_1, v_2\}$ is contained in $\mathrm{Span}\{v_1, v_2, v_3\}$.

d) True. Since A is 2×3 and the solution set to $Ax = 0$ is a plane, we have two free variables and thus A only has one pivot, so it cannot have a pivot in every row and thus $Ax = b$ will be inconsistent for some b in $\mathbf{R}^2$.

e) True: the column span of A contains all scalar multiples of $\begin{pmatrix} -2 \\ -1 \end{pmatrix}$, so it includes $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

Extra space for scratch work on problem 1

Problem 2.

[11 points]

You don't need to show work on (b). Parts (a)-(c) are 2 points each, and part (d) is 5 points.

a) Complete the following definition (be mathematically precise!):
Let $v_1, v_2, \ldots, v_p$ be vectors in $\mathbf{R}^n$. We say $\{v_1, v_2, \ldots, v_p\}$ is *linearly independent* if...

the equation $x_1 v_1 + \cdots + x_p v_p = 0$ has only the trivial solution

$$x_1 = \cdots = x_p = 0.$$

b) Suppose a homogeneous system of 4 linear equations in 3 unknowns corresponds to an augmented matrix with exactly two pivots. Then the solution set for the system is a:

(circle one answer) **point** $\boxed{\textbf{line}}$ **plane**

in:

(circle one answer) $\mathbf{R}$ $\mathbf{R}^2$ $\boxed{\mathbf{R}^3}$ $\mathbf{R}^4$.

c) Is there a 2×2 matrix A so that the solution set for the equation $Ax = 0$ is the line $x_1 = x_2 + 1$? If yes, write such an A. If no, justify why there is no such A.

No. The solution set to $Ax = 0$ must include the origin $x_1 = x_2 = 0$, which is not on the line $x_1 = x_2 + 1$.

d) Let $v_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$, $v_2 = \begin{pmatrix} -2 \\ -1 \\ 3 \end{pmatrix}$, $v_3 = \begin{pmatrix} 2 \\ -2 \\ c \end{pmatrix}$.

Find all values of c (if there are any) so that v_3 is a linear combination of v_1 and v_2.

We solve the equation $x_1 v_1 + x_2 v_2 = v_3$.

$$\left(\begin{array}{cc|c} 1 & -2 & 2 \\ 2 & -1 & -2 \\ -1 & 3 & c \end{array}\right) \xrightarrow[R_3 = R_3 + R_1]{R_2 = R_2 - 2R_1} \left(\begin{array}{cc|c} 1 & -2 & 2 \\ 0 & 3 & -6 \\ 0 & 1 & c+2 \end{array}\right) \xrightarrow[\text{then } R_3 = R_3 - R_2]{R_3 = R_3/3} \left(\begin{array}{cc|c} 1 & -2 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & c+4 \end{array}\right).$$

This is consistent if and only if $c + 4 = 0$, so $\boxed{c = -4}$.

Extra space for work on problem 2

Problem 3.

Parts (a) and (b) are unrelated. Part (a) is worth 5 points. Part (b) is worth 7 points.

a) Councilman Jamm loves the linear system of equations

$$2x - hy = k$$
$$4x + 10y = 5,$$

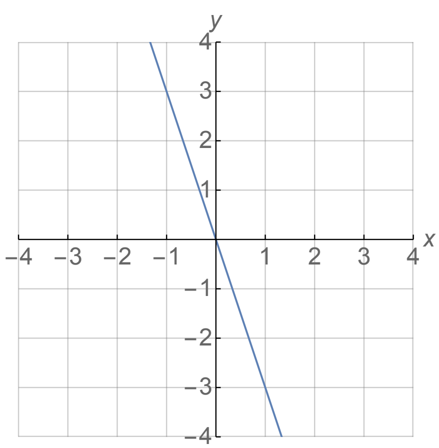
where h and k are real numbers. Find all values of h and k (if there are any) so that the system has infinitely many solutions.

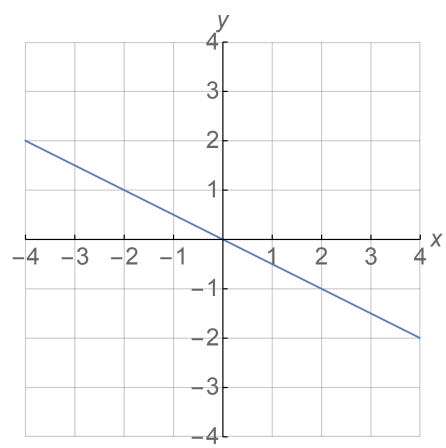
Solution: $\left(\begin{array}{cc|c} 2 & -h & k \\ 4 & 10 & 5 \end{array}\right) \xrightarrow{R_2=R_2-2R_1} \left(\begin{array}{cc|c} 2 & -h & k \\ 0 & 10+2h & 5-2k \end{array}\right)$.

The augmented matrix will have exactly one pivot (and no pivot in the final column) precisely when $10+2h=0$ but $5-2k=0$, so $h=-5$ and $k=\frac{5}{2}$.

b) Let $A = \begin{pmatrix} 1 & 2 \\ -3 & -6 \end{pmatrix}$. On the left graph, draw the span of the columns of A. On the right graph, draw the solution set for the equation $A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

The column span is Span $\left\{\begin{pmatrix} 1 \\ -3 \end{pmatrix}\right\}$ which is the line $y=-3x$. For the homogeneous solution set, $\left(\begin{array}{cc|c} 1 & 2 & 0 \\ -3 & -6 & 0 \end{array}\right) \xrightarrow{R_2=R_2+3R_1} \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right)$ so $x+2y=0$ yielding the line $y=-x/2$ which is Span $\left\{\begin{pmatrix} -2 \\ 1 \end{pmatrix}\right\}$.





Extra space for work on problem 3

Problem 4. [10 points]

Consider the following linear system of equations in the variables $x_1,\ x_2,\ x_3,\ x_4$:

$$\begin{aligned} x_1 - 2x_2 + x_4 &= 1 \\ x_1 - 2x_2 + x_3 + x_4 &= -2 \\ 3x_1 - 6x_2 + 2x_3 + 3x_4 &= -3. \end{aligned}$$

a) Write the augmented matrix corresponding to this system, and put the augmented matrix into RREF.

b) The system is consistent. Write the set of solutions to the system of equations in parametric vector form.

c) Write *one* vector that is not the zero vector and that is a solution for the corresponding homogeneous system of equations below. You do not need to show your work for this part.

$$\begin{aligned} x_1 - 2x_2 + x_4 &= 0 \\ x_1 - 2x_2 + x_3 + x_4 &= 0 \\ 3x_1 - 6x_2 + 2x_3 + 3x_4 &= 0. \end{aligned}$$

Solution.

a)

$$\left(\begin{array}{cccc|c} 1 & -2 & 0 & 1 & 1 \\ 1 & -2 & 1 & 1 & -2 \\ 3 & -6 & 2 & 3 & -3 \end{array}\right) \xrightarrow[R_3=R_3-2R_1]{R_2=R_2-R_1} \left(\begin{array}{cccc|c} 1 & -2 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 2 & 0 & -6 \end{array}\right) \xrightarrow{R_3=R_3-2R_2} \left(\begin{array}{cccc|c} 1 & -2 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

b) The RREF in part (a) shows that x_2 and x_4 are free and

$$x_1 = 1 + 2x_2 - x_4 \qquad x_2 = x_2 \qquad x_3 = -3 \qquad x_4 = x_4.$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 + 2x_2 - x_4 \\ x_2 \\ -3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -3 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

c) Any vector that lives in Span $\left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ is a homogeneous solution, so we just need to pick a nonzero vector there. For example $\begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ are correct answers.

Extra space for work on problem 4

Problem 5. [7 points]

Parts (a) and (b) are unrelated.

a) Write a single matrix A that satisfies all of the following conditions:

- The equation $Ax = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ is consistent, and the solution set is a line.
- The equation $Ax = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ is inconsistent.

Many answers possible. We need $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ in the column span and exactly one free variable, and $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ not in the column span. For example

$$A = \begin{pmatrix} 1 & 1 \\ -1 & -1 \\ 0 & 0 \end{pmatrix} \quad \text{or} \quad A = \begin{pmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{or} \quad A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

b) Suppose v_1, v_2, v_3, v_4 are vectors in $\mathbf{R}^4$. Which of the following statements must be true? Circle all that apply.

(i) If $\{v_1, v_2, v_3, v_4\}$ is linearly dependent, then v_4 is in Span$\{v_1, v_2, v_3\}$.

[(ii)] If $\{v_1, v_2, v_3\}$ is linearly independent, then $\{v_1, v_2\}$ is linearly independent.

[(iii)] If $\{v_1, v_2, v_3, v_4\}$ is linearly independent, then Span$\{v_1, v_2, v_3, v_4\} = \mathbf{R}^4$.

Extra space for work on problem 5