

# Eigenvectors and Eigenvalues

## Reminder

### Definition

Let  $A$  be an  $n \times n$  matrix.

1. An **eigenvector** of  $A$  is a nonzero vector  $v$  in  $\mathbf{R}^n$  such that  $Av = \lambda v$ , for some  $\lambda$  in  $\mathbf{R}$ .
2. An **eigenvalue** of  $A$  is a number  $\lambda$  in  $\mathbf{R}$  such that the equation  $Av = \lambda v$  has a nontrivial solution.
3. If  $\lambda$  is an eigenvalue of  $A$ , the  $\lambda$ -**eigenspace** is the solution set of  $(A - \lambda I_n)x = 0$ .

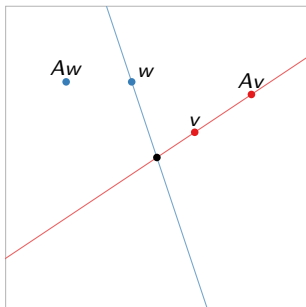
# Eigenspaces

## Geometry

### Eigenvectors, geometrically

An eigenvector of a matrix  $A$  is a nonzero vector  $v$  such that:

- ▶  $Av$  is a multiple of  $v$ , which means
- ▶  $Av$  is collinear with  $v$ , which means
- ▶  $Av$  and  $v$  are *on the same line through the origin*.



$v$  is an eigenvector

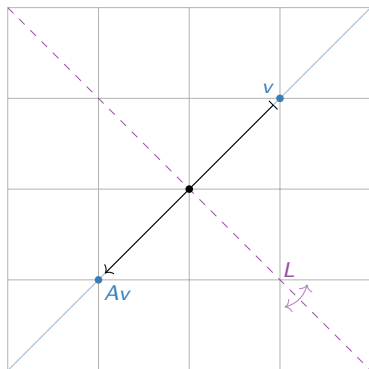
$w$  is not an eigenvector

# Eigenspaces

Geometry; example

Let  $T: \mathbf{R}^2 \rightarrow \mathbf{R}^2$  be reflection over the line  $L$  defined by  $y = -x$ , and let  $A$  be the matrix for  $T$ .

**Question:** What are the eigenvalues and eigenspaces of  $A$ ? No computations!



Does anyone see any eigenvectors  
(vectors that don't move off their line)?

$v$  is an eigenvector with eigenvalue  $-1$ .

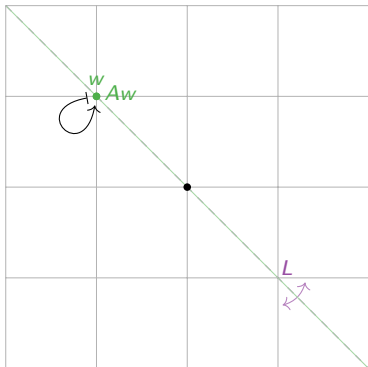
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$w$  is an eigenvector with eigenvalue 1.

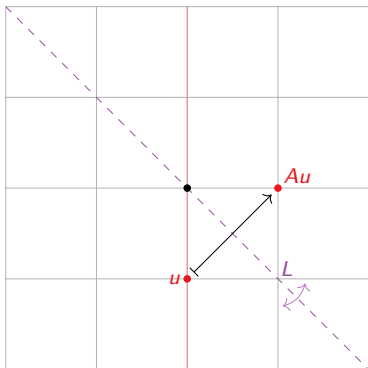
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$u$  is *not* an eigenvector.

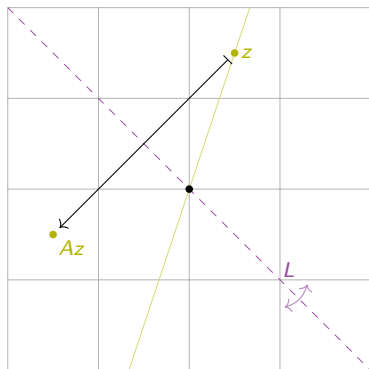
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Does anyone see any eigenvectors  
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Neither is  $z$ .

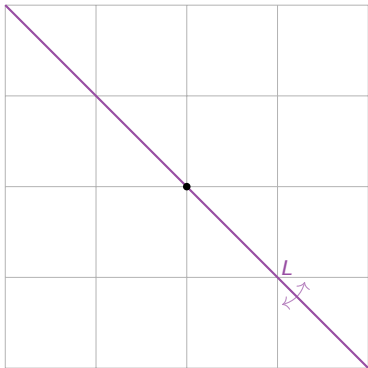
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The 1-eigenspace is  $L$   
(all the vectors  $x$  where  $Ax = x$ ).

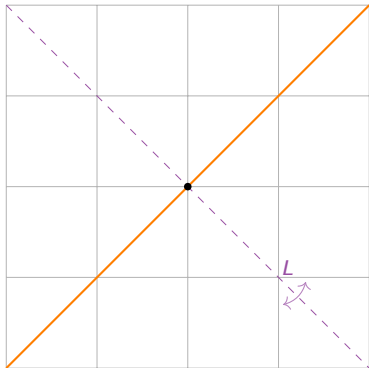
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# Eigenspaces

Geometry; example

Let  $T: \mathbf{R}^2 \rightarrow \mathbf{R}^2$  be reflection over the line  $L$  defined by  $y = -x$ , and let  $A$  be the matrix for  $T$ .

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The  $(-1)$ -eigenspace is **the line  $y = x$**   
(all the vectors  $x$  where  $Ax = -x$ ).

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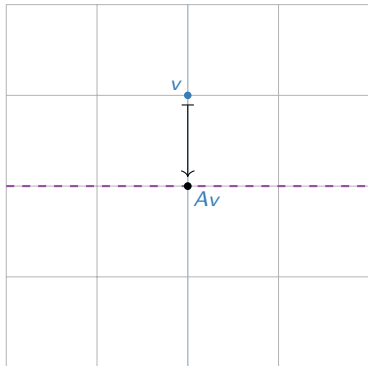


# Eigenspaces

Geometry; example

Let  $T: \mathbf{R}^2 \rightarrow \mathbf{R}^2$  be the vertical projection onto the  $x$ -axis, and let  $A$  be the matrix for  $T$ .

**Question:** What are the eigenvalues and eigenspaces of  $A$ ? No computations!



Does anyone see any eigenvectors (vectors that don't move off their line)?

$v$  is an eigenvector with eigenvalue 0.

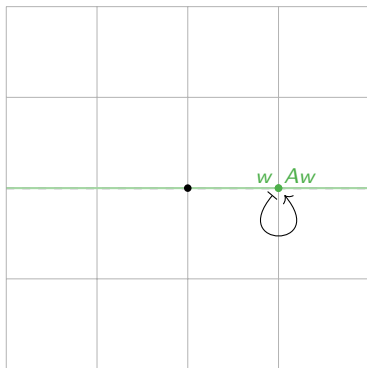
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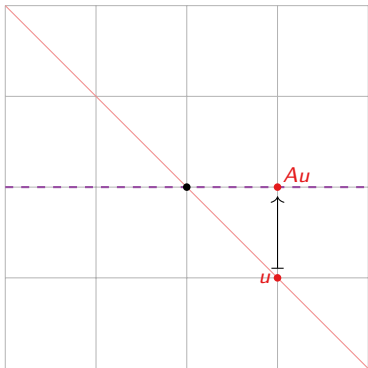
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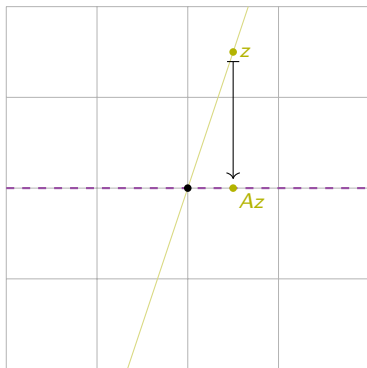
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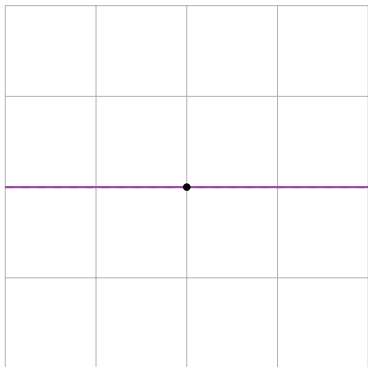
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The 1-eigenspace is **the  $x$ -axis** (all the vectors  $x$  where  $Ax = x$ ).

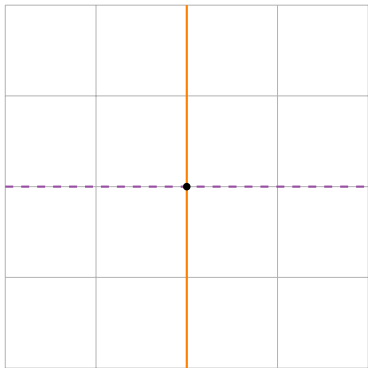
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The 0-eigenspace is **the  $y$ -axis**  
(all the vectors  $x$  where  $Ax = 0x$ ).

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# Eigenspaces

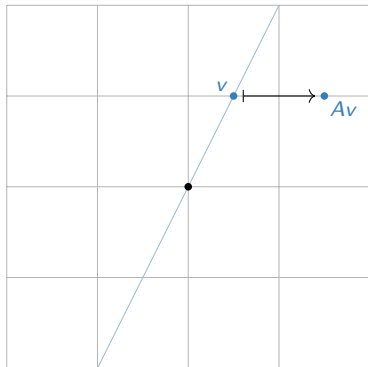
Geometry; example

Let

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

so  $T(x) = Ax$  is a shear in the  $x$ -direction.

**Question:** What are the eigenvalues and eigenspaces of  $A$ ? No computations!



Does anyone see any eigenvectors (vectors that don't move off their line)?

Vectors  $v$  above the  $x$ -axis are moved right but not up...  
so they're not eigenvectors.

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# Eigenspaces

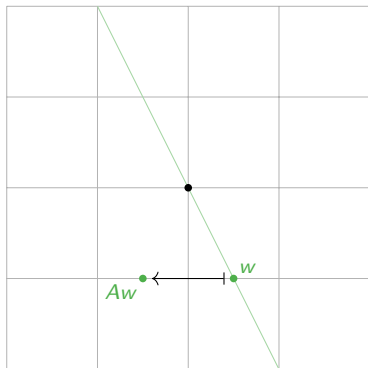
Geometry; example

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Does anyone see any eigenvectors  
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Vectors  $w$  below the  $x$ -axis are moved  
left but not down...  
so they're not eigenvectors

[interactive]



# Eigenspaces

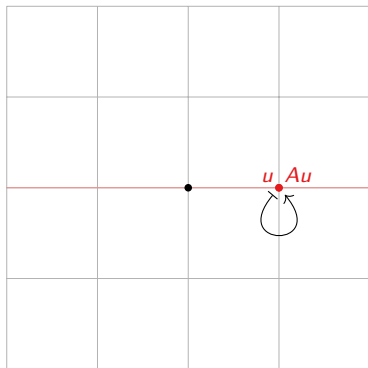
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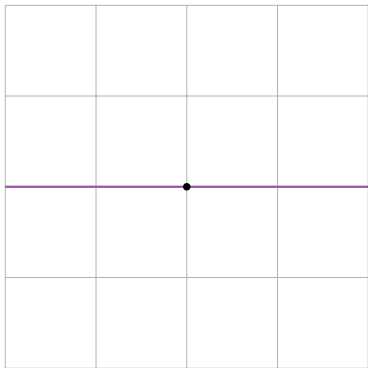
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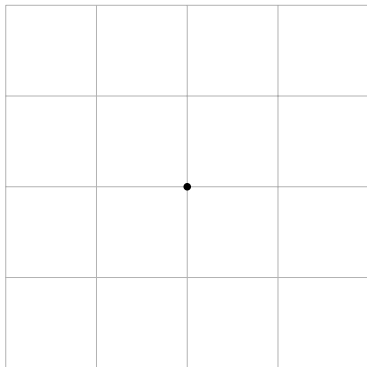
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**Question:** What are the eigenvalues and eigenspaces of  $A$ ? No computations!



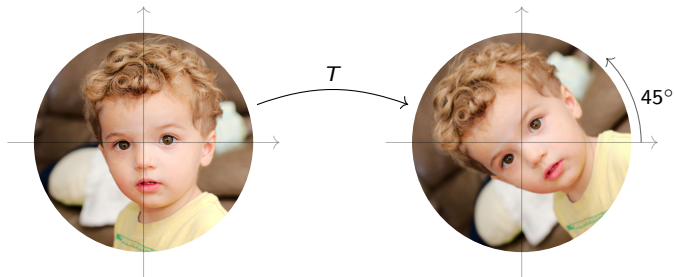
Does anyone see any eigenvectors  
(vectors that don't move off their line)?

There are no other eigenvectors.

[interactive]

## Poll

Let  $T: \mathbf{R}^2 \rightarrow \mathbf{R}^2$  be counterclockwise rotation by  $45^\circ$ , and let  $A$  be the matrix for  $T$ .



Poll

Find an eigenvector of  $A$  without doing any computations.

- A.** Okay.      **B.** No way.

**Answer:** **B.** No way. There are no eigenvectors!

[interactive]

## Section 5.2

### The Characteristic Polynomial

## The Characteristic Polynomial

Let  $A$  be a square matrix.

$$\begin{aligned}\lambda \text{ is an eigenvalue of } A &\iff Ax = \lambda x \text{ has a nontrivial solution} \\ &\iff (A - \lambda I)x = 0 \text{ has a nontrivial solution} \\ &\iff A - \lambda I \text{ is not invertible} \\ &\iff \det(A - \lambda I) = 0.\end{aligned}$$

This gives us a way to compute the eigenvalues of  $A$ .

### Definition

Let  $A$  be a square matrix. The **characteristic polynomial** of  $A$  is

$$f(\lambda) = \det(A - \lambda I).$$

The **characteristic equation** of  $A$  is the equation

$$f(\lambda) = \det(A - \lambda I) = 0.$$

#### Important

The eigenvalues of  $A$  are the roots of the characteristic polynomial  $f(\lambda) = \det(A - \lambda I)$ .

# The Characteristic Polynomial

## Example

**Question:** What are the eigenvalues of

$$A = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}?$$

**Answer:** First we find the characteristic polynomial:

$$\begin{aligned} f(\lambda) &= \det(A - \lambda I) = \det \left[ \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right] = \det \begin{pmatrix} 5 - \lambda & 2 \\ 2 & 1 - \lambda \end{pmatrix} \\ &= (5 - \lambda)(1 - \lambda) - 2 \cdot 2 \\ &= \lambda^2 - 6\lambda + 1. \end{aligned}$$

The eigenvalues are the roots of the characteristic polynomial, which we can find using the quadratic formula:

$$\lambda = \frac{6 \pm \sqrt{36 - 4}}{2} = 3 \pm 2\sqrt{2}.$$

# The Characteristic Polynomial

## Example

**Question:** What is the characteristic polynomial of

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}?$$

**Answer:**

$$\begin{aligned} f(\lambda) &= \det(A - \lambda I) = \det \begin{pmatrix} a - \lambda & b \\ c & d - \lambda \end{pmatrix} = (a - \lambda)(d - \lambda) - bc \\ &= \lambda^2 - (a + d)\lambda + (ad - bc) \end{aligned}$$

What do you notice about  $f(\lambda)$ ?

- ▶ The constant term is  $\det(A)$ , which is zero if and only if  $\lambda = 0$  is a root.
- ▶ The linear term  $-(a + d)$  is the negative of the sum of the diagonal entries of  $A$ .

## Definition

The **trace** of a square matrix  $A$  is  $\text{Tr}(A) =$  sum of the diagonal entries of  $A$ .

### Shortcut

The characteristic polynomial of a  $2 \times 2$  matrix  $A$  is

$$f(\lambda) = \lambda^2 - \text{Tr}(A)\lambda + \det(A).$$



# The Characteristic Polynomial

## Example

**Question:** What are the eigenvalues of the rabbit population matrix

$$A = \begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix}?$$

**Answer:** First we find the characteristic polynomial:

$$\begin{aligned} f(\lambda) &= \det(A - \lambda I) = \det \begin{pmatrix} -\lambda & 6 & 8 \\ \frac{1}{2} & -\lambda & 0 \\ 0 & \frac{1}{2} & -\lambda \end{pmatrix} \\ &= 8 \left( \frac{1}{4} - 0 \cdot -\lambda \right) - \lambda \left( \lambda^2 - 6 \cdot \frac{1}{2} \right) \\ &= -\lambda^3 + 3\lambda + 2. \end{aligned}$$

We know from before that one eigenvalue is  $\lambda = 2$ : indeed,  $f(2) = -8 + 6 + 2 = 0$ . Doing polynomial long division, we get:

$$\frac{-\lambda^3 + 3\lambda + 2}{\lambda - 2} = -\lambda^2 - 2\lambda - 1 = -(\lambda + 1)^2.$$

Hence  $\lambda = -1$  is also an eigenvalue.

## Factoring the Characteristic Polynomial

It's easy to factor quadratic polynomials:

$$x^2 + bx + c = 0 \implies x = \frac{-b \pm \sqrt{b^2 - 4c}}{2}.$$

It's less easy to factor cubics, quartics, and so on:

$$x^3 + bx^2 + cx + d = 0 \implies x = ???$$

$$x^4 + bx^3 + cx^2 + dx + e = 0 \implies x = ???$$

Read about factoring polynomials by hand in §5.2.

## Summary

We did two different things today.

First we talked about the geometry of eigenvalues and eigenvectors:

- ▶ Eigenvectors are vectors  $v$  such that  $v$  and  $Av$  are on the same line through the origin.
- ▶ You can pick out the eigenvectors geometrically if you have a picture of the associated transformation.

Then we talked about characteristic polynomials:

- ▶ We learned to find the eigenvalues of a matrix by computing the roots of the characteristic polynomial  $p(\lambda) = \det(A - \lambda I)$ .
- ▶ For a  $2 \times 2$  matrix  $A$ , the characteristic polynomial is just

$$p(\lambda) = \lambda^2 - \text{Tr}(A)\lambda + \det(A).$$