Chapter 5

Eigenvalues and Eigenvectors

Section 5.1

Eigenvalues and Eigenvectors

A Biology Question

Motivation

In a population of rabbits:

1. half of the newborn rabbits survive their first year;
2. of those, half survive their second year;
3. their maximum life span is three years;
4. rabbits have $0, 6, 8$ baby rabbits in their three years, respectively.

If you know the population one year, what is the population the next year?

$$\begin{aligned} f_n &= \text{first-year rabbits in year } n \\ s_n &= \text{second-year rabbits in year } n \\ t_n &= \text{third-year rabbits in year } n \end{aligned}$$

The rules say:

$$\begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} f_n \\ s_n \\ t_n \end{pmatrix} = \begin{pmatrix} f_{n+1} \\ s_{n+1} \\ t_{n+1} \end{pmatrix}.$$

Let $A = \begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix}$ and $v_n = \begin{pmatrix} f_n \\ s_n \\ t_n \end{pmatrix}$. Then $\boxed{Av_n = v_{n+1}.}$ ⟵ **difference equation**

A Biology Question

Continued

If you know v_0, what is v_{10}?

$$v_{10} = Av_9 = AAv_8 = \cdots = A^{10}v_0.$$

This makes it easy to compute examples by computer: [interactive]

v_0	v_{10}	v_{11}
$\begin{pmatrix} 3 \\ 7 \\ 9 \end{pmatrix}$	$\begin{pmatrix} 30189 \\ 7761 \\ 1844 \end{pmatrix}$	$\begin{pmatrix} 61316 \\ 15095 \\ 3881 \end{pmatrix}$
$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$	$\begin{pmatrix} 9459 \\ 2434 \\ 577 \end{pmatrix}$	$\begin{pmatrix} 19222 \\ 4729 \\ 1217 \end{pmatrix}$
$\begin{pmatrix} 4 \\ 7 \\ 8 \end{pmatrix}$	$\begin{pmatrix} 28856 \\ 7405 \\ 1765 \end{pmatrix}$	$\begin{pmatrix} 58550 \\ 14428 \\ 3703 \end{pmatrix}$

What do you notice about these numbers?

1. Eventually, each segment of the population doubles every year: $Av_n = v_{n+1} = 2v_n$.
2. The ratios get close to $(16 : 4 : 1)$:

$$v_n = (\text{scalar}) \cdot \begin{pmatrix} 16 \\ 4 \\ 1 \end{pmatrix}.$$

Translation: 2 is an eigenvalue, and $\begin{pmatrix} 16 \\ 4 \\ 1 \end{pmatrix}$ is an eigenvector!

Eigenvectors and Eigenvalues

Definition

Let A be an $n \times n$ matrix.

Eigenvalues and eigenvectors are only for square matrices.

1. An **eigenvector** of A is a *nonzero* vector v in $\mathbf{R}^n$ such that $Av = \lambda v$, for some λ in $\mathbf{R}$. In other words, Av is a multiple of v.
2. An **eigenvalue** of A is a number λ in $\mathbf{R}$ such that the equation $Av = \lambda v$ has a *nontrivial* solution.

If $Av = \lambda v$ for $v \neq 0$, we say λ is the **eigenvalue for** v, and v is an **eigenvector for** λ.

Note: Eigenvectors are by definition nonzero. Eigenvalues may be equal to zero.

This is the most important definition in the course.

Verifying Eigenvectors

Example

$$A = \begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \qquad v = \begin{pmatrix} 16 \\ 4 \\ 1 \end{pmatrix}$$

Multiply:

$$Av = \begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} 16 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 32 \\ 8 \\ 2 \end{pmatrix} = 2v$$

Hence v is an eigenvector of A, with eigenvalue $\lambda = 2$.

Example

$$A = \begin{pmatrix} 2 & 2 \\ -4 & 8 \end{pmatrix} \qquad v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Multiply:

$$Av = \begin{pmatrix} 2 & 2 \\ -4 & 8 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix} = 4v$$

Hence v is an eigenvector of A, with eigenvalue $\lambda = 4$.

Poll

Poll

Which of the vectors

A. $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ B. $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ C. $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ D. $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ E. $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

are eigenvectors of the matrix $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$?

What are the eigenvalues?

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

eigenvector with eigenvalue 2

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 0 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

eigenvector with eigenvalue 0

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = 0 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

eigenvector with eigenvalue 0

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$$

not an eigenvector

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

is never an eigenvector

Verifying Eigenvalues

Question: Is $\lambda = 3$ an eigenvalue of $A = \begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix}$?

In other words, does $Av = 3v$ have a nontrivial solution?

. . . does $Av - 3v = 0$ have a nontrivial solution?

. . . does $(A - 3I)v = 0$ have a nontrivial solution?

We know how to answer that! Row reduction!

$$A - 3I = \begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix} - 3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & -4 \\ -1 & -4 \end{pmatrix}$$

Row reduce:

$$\begin{pmatrix} -1 & -4 \\ -1 & -4 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 4 \\ 0 & 0 \end{pmatrix}$$

Parametric form: $x = -4y$; parametric vector form: $\begin{pmatrix} x \\ y \end{pmatrix} = y\begin{pmatrix} -4 \\ 1 \end{pmatrix}$.

Does there exist an eigenvector with eigenvalue $\lambda = 3$? Yes! Any nonzero multiple of $\begin{pmatrix} -4 \\ 1 \end{pmatrix}$. Check:

$$\begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} -4 \\ 1 \end{pmatrix} = \begin{pmatrix} -12 \\ 3 \end{pmatrix} = 3\begin{pmatrix} -4 \\ 1 \end{pmatrix}.$$

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Eigenspaces

Definition

Let A be an $n \times n$ matrix and let λ be an eigenvalue of A. The λ-**eigenspace** of A is the set of all eigenvectors of A with eigenvalue λ, plus the zero vector:

$$\begin{aligned}\lambda\text{-eigenspace} &= \bigl\{v \text{ in } \mathbf{R}^n \mid Av = \lambda v\bigr\} \\ &= \bigl\{v \text{ in } \mathbf{R}^n \mid (A - \lambda I)v = 0\bigr\} \\ &= \operatorname{Nul}\bigl(A - \lambda I\bigr).\end{aligned}$$

Since the λ-eigenspace is a null space, it is a *subspace* of $\mathbf{R}^n$.

How do you find a basis for the λ-eigenspace? Parametric vector form!

Eigenspaces

Example

Find a basis for the $\underset{\lambda}{3}$-eigenspace of

$$A = \begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix}.$$

We have to solve the matrix equation $A - 3I_2 = 0$.

$$A - 3I_2 = \begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix} - 3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & -4 \\ -1 & -4 \end{pmatrix}$$

$$\overset{\text{RREF}}{\rightsquigarrow} \begin{pmatrix} 1 & 4 \\ 0 & 0 \end{pmatrix}$$

$$\overset{\text{parametric form}}{\rightsquigarrow} x = -4y$$

$$\overset{\text{parametric vector form}}{\rightsquigarrow} \begin{pmatrix} x \\ y \end{pmatrix} = y\begin{pmatrix} -4 \\ 1 \end{pmatrix}$$

$$\overset{\text{basis}}{\rightsquigarrow} \left\{ \begin{pmatrix} -4 \\ 1 \end{pmatrix} \right\}.$$

Eigenspaces

Example

Find a basis for the $\underset{\lambda}{2}$-eigenspace of

$$A = \begin{pmatrix} 7/2 & 0 & 3 \\ -3/2 & 2 & -3 \\ -3/2 & 0 & -1 \end{pmatrix}.$$

$$A - 2I = \begin{pmatrix} \frac{3}{2} & 0 & 3 \\ -\frac{3}{2} & 0 & -3 \\ -\frac{3}{2} & 0 & -3 \end{pmatrix} \overset{\text{row reduce}}{\rightsquigarrow} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\overset{\text{parametric form}}{\rightsquigarrow} \quad x = -2z$$

$$\overset{\text{parametric vector form}}{\rightsquigarrow} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = y \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$$

$$\overset{\text{basis}}{\rightsquigarrow} \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Eigenspaces

Example

Find a basis for the $\frac{1}{2}$-eigenspace of

$$A = \begin{pmatrix} 7/2 & 0 & 3 \\ -3/2 & 2 & -3 \\ -3/2 & 0 & -1 \end{pmatrix}.$$

$$A - \frac{1}{2}I = \begin{pmatrix} 3 & 0 & 3 \\ -\frac{3}{2} & \frac{3}{2} & -3 \\ -\frac{3}{2} & 0 & -\frac{3}{2} \end{pmatrix} \overset{\text{row reduce}}{\rightsquigarrow} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\overset{\text{parametric form}}{\rightsquigarrow} \begin{cases} x = -z \\ y = z \end{cases}$$

$$\overset{\text{parametric vector form}}{\rightsquigarrow} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = z \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

$$\overset{\text{basis}}{\rightsquigarrow} \left\{ \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

Eigenspaces

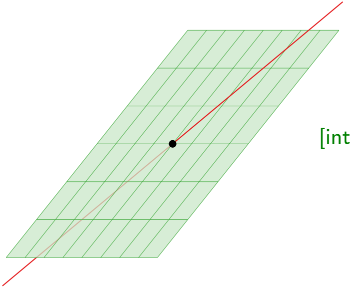
Example: picture

$$A = \begin{pmatrix} 7/2 & 0 & 3 \\ -3/2 & 2 & -3 \\ -3/2 & 0 & -1 \end{pmatrix}.$$

We computed bases for the 2-eigenspace and the 1/2-eigenspace:

$$\text{2-eigenspace: } \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \right\} \qquad \frac{1}{2}\text{-eigenspace: } \left\{ \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

Hence the 2-eigenspace is a plane and the 1/2-eigenspace is a line.



[interactive]

Eigenspaces

Summary

Let A be an $n \times n$ matrix and let λ be a number.

1. λ is an eigenvalue of A if and only if $(A - \lambda I)x = 0$ has a nontrivial solution, if and only if $\mathrm{Nul}(A - \lambda I) \neq \{0\}$.
2. In this case, finding a basis for the λ-eigenspace of A means finding a basis for $\mathrm{Nul}(A - \lambda I)$ as usual, i.e. by finding the parametric vector form for the general solution to $(A - \lambda I)x = 0$.
3. The eigenvectors with eigenvalue λ are the nonzero elements of $\mathrm{Nul}(A - \lambda I)$, i.e. the nontrivial solutions to $(A - \lambda I)x = 0$.

The Eigenvalues of a Triangular Matrix are the Diagonal Entries

We've seen that finding eigenvectors for a given eigenvalue is a row reduction problem.

Finding all of the eigenvalues of a matrix *is not a row reduction problem!* We'll see how to do it in general next time. For now:

Fact: The eigenvalues of a triangular matrix are the diagonal entries.

Why? $\mathrm{Nul}(A - \lambda I) \neq \{0\}$ if and only if $A - \lambda I$ is not invertible, if and only if $\det(A - \lambda I) = 0$.

$$\begin{pmatrix} 3 & 4 & 1 & 2 \\ 0 & -1 & -2 & 7 \\ 0 & 0 & 8 & 12 \\ 0 & 0 & 0 & -3 \end{pmatrix} - \lambda I_4 = \begin{pmatrix} 3-\lambda & 4 & 1 & 2 \\ 0 & -1-\lambda & -2 & 7 \\ 0 & 0 & 8-\lambda & 12 \\ 0 & 0 & 0 & -3-\lambda \end{pmatrix}.$$

The determinant is $(3-\lambda)(-1-\lambda)(8-\lambda)(-3-\lambda)$, which is zero exactly when $\lambda = 3, -1, 8,$ or -3.

A Matrix is Invertible if and only if Zero is not an Eigenvalue

Fact: A is invertible if and only if 0 is not an eigenvalue of A.

Why?

$$\begin{aligned} 0 \text{ is an eigenvalue of } A &\iff Ax = 0x \text{ has a nontrivial solution} \\ &\iff Ax = 0 \text{ has a nontrivial solution} \\ &\iff A \text{ is not invertible.} \end{aligned}$$

invertible matrix theorem

Eigenvectors with Distinct Eigenvalues are Linearly Independent

Fact: If $v_1, v_2, \ldots, v_k$ are eigenvectors of A with *distinct* eigenvalues $\lambda_1, \ldots, \lambda_k$, then $\{v_1, v_2, \ldots, v_k\}$ is linearly independent.

Why? If $k = 2$, this says v_2 can't lie on the line through v_1.

But the line through v_1 is contained in the λ_1-eigenspace, and v_2 does not have eigenvalue λ_1.

In general: see §6.1 (or work it out for yourself; it's not too hard).

Consequence: An $n \times n$ matrix has at most n distinct eigenvalues.

The Invertible Matrix Theorem

Addenda

We have a couple of new ways of saying “A is invertible” now:

The Invertible Matrix Theorem

Let A be a square $n \times n$ matrix, and let $T \colon \mathbf{R}^n \to \mathbf{R}^n$ be the linear transformation $T(x) = Ax$. The following statements are equivalent.

1. A is invertible.
2. T is invertible.
3. The reduced row echelon form of A is I_n.
4. A has n pivots.
5. $Ax = 0$ has no solutions other than the trivial one.
6. $\mathrm{Nul}(A) = \{0\}$.
7. $\mathrm{nullity}(A) = 0$.
8. The columns of A are linearly independent.
9. The columns of A form a basis for $\mathbf{R}^n$.
10. T is one-to-one.
11. $Ax = b$ is consistent for all b in $\mathbf{R}^n$.
12. $Ax = b$ has a unique solution for each b in $\mathbf{R}^n$.
13. The columns of A span $\mathbf{R}^n$.
14. $\mathrm{Col}\, A = \mathbf{R}^m$.
15. $\dim \mathrm{Col}\, A = m$.
16. $\mathrm{rank}\, A = m$.
17. T is onto.
18. There exists a matrix B such that $AB = I_n$.
19. There exists a matrix B such that $BA = I_n$.
20. The determinant of A is *not* equal to zero.
21. The number 0 is *not* an eigenvalue of A.

Summary

- **Eigenvectors** and **eigenvalues** are the most important concepts in this course.
- Eigenvectors are by definition nonzero; eigenvalues may be zero.
- The eigenvalues of a triangular matrix are the diagonal entries.
- A matrix is invertible if and only if zero is not an eigenvalue.
- Eigenvectors with distinct eigenvalues are linearly independent.
- The λ-eigenspace is the set of all λ-eigenvectors, plus the zero vector.
- You can compute a basis for the λ-eigenspace by finding the parametric vector form of the solutions of $(A - \lambda I_n)x = 0$.