

## Section 3.2

### One-to-one and Onto Transformations

# Matrix Transformations

## Reminder

**Recall:** Let  $A$  be an  $m \times n$  matrix. The **matrix transformation** associated to  $A$  is the transformation

$$T: \mathbf{R}^n \longrightarrow \mathbf{R}^m \quad \text{defined by} \quad T(x) = Ax.$$

- ▶ The *domain* of  $T$  is  $\mathbf{R}^n$ , which is the number of *columns* of  $A$ .
- ▶ The *codomain* of  $T$  is  $\mathbf{R}^m$ , which is the number of *rows* of  $A$ .
- ▶ The *range* of  $T$  is the set of all images of  $T$ :

$$T(x) = Ax = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x_1 v_1 + x_2 v_2 + \cdots + x_n v_n.$$

This is the *column space* of  $A$ . It is a span of vectors in the codomain.

# Matrix Transformations

## Example

Let  $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$  and let  $T(x) = Ax$ , so  $T: \mathbf{R}^2 \rightarrow \mathbf{R}^3$ .

► If  $u = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$  then  $T(u) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \\ 7 \end{pmatrix}$ .

► Let  $b = \begin{pmatrix} 7 \\ 5 \\ 7 \end{pmatrix}$ . Find  $v$  in  $\mathbf{R}^2$  such that  $T(v) = b$ . Is there more than one?

We want to find  $v$  such that  $T(v) = Av = b$ . We know how to do that:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} v = \begin{pmatrix} 7 \\ 5 \\ 7 \end{pmatrix} \xrightarrow{\text{augmented matrix}} \left( \begin{array}{cc|c} 1 & 1 & 7 \\ 0 & 1 & 5 \\ 1 & 1 & 7 \end{array} \right) \xrightarrow{\text{reduce}} \left( \begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{array} \right).$$

This gives  $x = 2$  and  $y = 5$ , or  $v = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$  (unique). In other words,

$$T(v) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \\ 7 \end{pmatrix}.$$

# Matrix Transformations

## Example, continued

Let  $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$  and let  $T(x) = Ax$ , so  $T: \mathbf{R}^2 \rightarrow \mathbf{R}^3$ .

- ▶ Is there any  $c$  in  $\mathbf{R}^3$  such that there is more than one  $v$  in  $\mathbf{R}^2$  with  $T(v) = c$ ?

**Translation:** is there any  $c$  in  $\mathbf{R}^3$  such that the solution set of  $Ax = c$  has more than one vector  $v$  in it?

The solution set of  $Ax = c$  is a translate of the solution set of  $Ax = b$  (from before), which has one vector in it. So the solution set to  $Ax = c$  has only one vector. So no!

- ▶ Find  $c$  such that there is *no*  $v$  with  $T(v) = c$ .

**Translation:** Find  $c$  such that  $Ax = c$  is inconsistent.

**Translation:** Find  $c$  not in the column space of  $A$  (i.e., the range of  $T$ ).

We could draw a picture, or notice that if  $c = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ , then our matrix equation translates into

$$x + y = 1 \quad y = 2 \quad x + y = 3,$$

which is obviously inconsistent.

# Matrix Transformations

## Non-Example

**Note:** All of these questions are questions about *the transformation*  $T$ ; it still makes sense to ask them in the absence of the matrix  $A$ .

The fact that  $T$  comes from a matrix means that these questions translate into questions about a matrix, which we know how to do.

**Non-example:**  $T: \mathbf{R}^2 \rightarrow \mathbf{R}^3$       $T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \sin x \\ xy \\ \cos y \end{pmatrix}$

**Question:** Is there any  $c$  in  $\mathbf{R}^3$  such that there is more than one  $v$  in  $\mathbf{R}^2$  with  $T(v) = c$ ?

Note the question still makes sense, although  $T$  has no hope of being a matrix transformation.

By the way,

$$T \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \sin 0 \\ 0 \cdot 0 \\ \cos 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \sin \pi \\ 0 \cdot \pi \\ \cos 0 \end{pmatrix} = T \begin{pmatrix} \pi \\ 0 \end{pmatrix},$$

so the answer is yes.

## Questions About Transformations

Today we will focus on two important questions one can ask about a transformation  $T: \mathbf{R}^n \rightarrow \mathbf{R}^m$ :

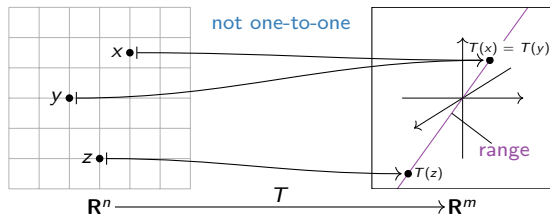
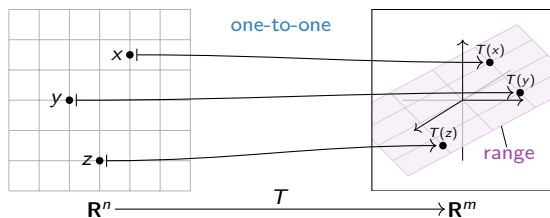
- ▶ Do there exist distinct vectors  $x, y$  in  $\mathbf{R}^n$  such that  $T(x) = T(y)$ ?
- ▶ For every vector  $v$  in  $\mathbf{R}^m$ , does there exist a vector  $x$  in  $\mathbf{R}^n$  such that  $T(x) = v$ ?

These are subtle because of the multiple *quantifiers* involved (“for every”, “there exists”).

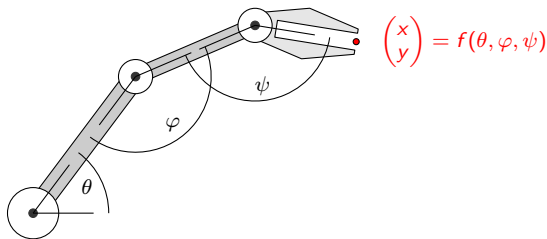
# One-to-one Transformations

## Definition

A transformation  $T: \mathbf{R}^n \rightarrow \mathbf{R}^m$  is **one-to-one** (or **into**, or **injective**) if different vectors in  $\mathbf{R}^n$  map to different vectors in  $\mathbf{R}^m$ . In other words, for every  $b$  in  $\mathbf{R}^m$ , the equation  $T(x) = b$  has *at most one* solution  $x$ . Or, different inputs have different outputs. Note that *not* one-to-one means at least two different vectors in  $\mathbf{R}^n$  have the same image.



Consider the robot hand transformation from last lecture:



Define  $f: \mathbf{R}^3 \rightarrow \mathbf{R}^2$  by:

$f(\theta, \varphi, \psi) =$  position of the hand at joint angles  $\theta, \varphi, \psi$ .

Poll

Is  $f$  one-to-one?

**No:** there is more than one way to move the hand to the same point.



## Characterization of One-to-One Matrix Transformations

### Theorem

Let  $T: \mathbf{R}^n \rightarrow \mathbf{R}^m$  be a matrix transformation with matrix  $A$ . Then the following are equivalent:

- ▶  $T$  is one-to-one.
- ▶ For each  $b$  in  $\mathbf{R}^m$ , the equation  $T(x) = b$  has at most one solution.
- ▶ For each  $b$  in  $\mathbf{R}^m$ , the equation  $Ax = b$  has a unique solution or is inconsistent.
- ▶  $Ax = 0$  has a unique solution.
- ▶ The columns of  $A$  are linearly independent.
- ▶  $A$  has a pivot in every column.

### Question

If  $T: \mathbf{R}^n \rightarrow \mathbf{R}^m$  is one-to-one, what can we say about the relative sizes of  $n$  and  $m$ ?

**Answer:**  $T$  corresponds to an  $m \times n$  matrix  $A$ . In order for  $A$  to have a pivot in every column, it must have *at least as many* rows as columns:  $n \leq m$ .

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

For instance,  $\mathbf{R}^3$  is “too big” to map *into*  $\mathbf{R}^2$ .

## One-to-One Transformations

### Example

Define

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \quad T(x) = Ax,$$

so  $T: \mathbf{R}^2 \rightarrow \mathbf{R}^3$ . Is  $T$  one-to-one?

The reduced row echelon form of  $A$  is

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

which has a pivot in every column. Hence  $T$  is one-to-one.

[interactive]

# One-to-One Transformations

## Non-Example

Define

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad T(x) = Ax,$$

so  $T: \mathbf{R}^3 \rightarrow \mathbf{R}^2$ . Is  $T$  one-to-one? If not, find two different vectors  $x, y$  such that  $T(x) = T(y)$ .

The reduced row echelon form of  $A$  is

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}$$

which does not have a pivot in every column. Hence  $A$  is not one-to-one. In particular,  $Ax = 0$  has nontrivial solutions. The parametric form of the solutions of  $Ax = 0$  are

$$\begin{array}{rcl} x & -z = 0 & \implies x = z \\ y + z = 0 & & y = -z. \end{array}$$

Taking  $z = 1$  gives

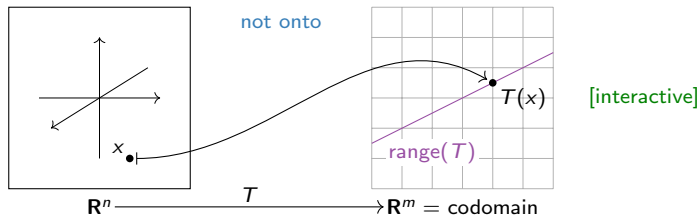
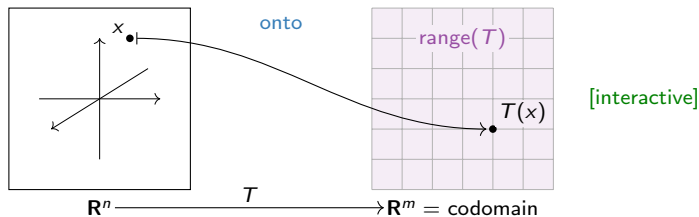
$$T \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = 0 = T \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

[interactive]

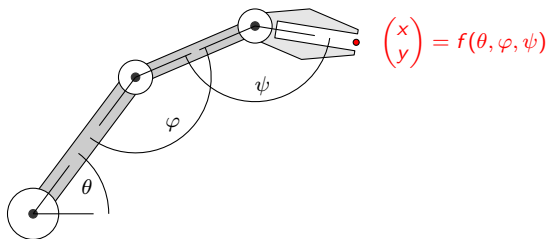
# Onto Transformations

## Definition

A transformation  $T: \mathbf{R}^n \rightarrow \mathbf{R}^m$  is **onto** (or **surjective**) if the range of  $T$  is equal to  $\mathbf{R}^m$  (its codomain). In other words, for every  $b$  in  $\mathbf{R}^m$ , the equation  $T(x) = b$  has at least one solution. Or, every possible output has an input. Note that *not* onto means there is some  $b$  in  $\mathbf{R}^m$  which is not the image of any  $x$  in  $\mathbf{R}^n$ .



Consider the robot hand transformation again:



Define  $f: \mathbf{R}^3 \rightarrow \mathbf{R}^2$  by:

$f(\theta, \varphi, \psi) =$  position of the hand at joint angles  $\theta, \varphi, \psi$ .

Poll

Is  $f$  onto?

**No:** it can't reach points that are far away.

## Characterization of Onto Matrix Transformations

### Theorem

Let  $T: \mathbf{R}^n \rightarrow \mathbf{R}^m$  be a matrix transformation with matrix  $A$ . Then the following are equivalent:

- ▶  $T$  is onto
- ▶  $T(x) = b$  has a solution for every  $b$  in  $\mathbf{R}^m$
- ▶  $Ax = b$  is consistent for every  $b$  in  $\mathbf{R}^m$
- ▶ The columns of  $A$  span  $\mathbf{R}^m$
- ▶  $A$  has a pivot in every row

### Question

If  $T: \mathbf{R}^n \rightarrow \mathbf{R}^m$  is onto, what can we say about the relative sizes of  $n$  and  $m$ ?

**Answer:**  $T$  corresponds to an  $m \times n$  matrix  $A$ . In order for  $A$  to have a pivot in every row, it must have *at least as many* columns as rows:  $m \leq n$ .

$$\begin{pmatrix} 1 & 0 & * & 0 & * \\ 0 & 1 & * & 0 & * \\ 0 & 0 & 0 & 1 & * \end{pmatrix}$$

For instance,  $\mathbf{R}^2$  is “too small” to map *onto*  $\mathbf{R}^3$ .

## Onto Transformations

### Example

Define

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad T(x) = Ax,$$

so  $T: \mathbf{R}^3 \rightarrow \mathbf{R}^2$ . Is  $T$  onto?

The reduced row echelon form of  $A$  is

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}$$

which has a pivot in every row. Hence  $T$  is onto.

Note that  $T$  is *onto* but not *one-to-one*.

[interactive]

# Onto Transformations

## Non-Example

Define

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \quad T(x) = Ax,$$

so  $T: \mathbf{R}^2 \rightarrow \mathbf{R}^3$ . Is  $T$  onto? If not, find a vector  $v$  in  $\mathbf{R}^3$  such that there does not exist any  $x$  in  $\mathbf{R}^2$  with  $T(x) = v$ .

The reduced row echelon form of  $A$  is

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

which does not have a pivot in every row. Hence  $A$  is not onto.

In order to find a vector  $v$  not in the range, we notice that  $T\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \\ a \end{pmatrix}$ . In particular, the  $x$ - and  $z$ -coordinates are the same for every vector in the range, so for example,  $v = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$  is not in the range.

Note that  $T$  is *one-to-one* but not *onto*.

[interactive]



## One-to-One and Onto Transformations

### Non-Example

Define

$$A = \begin{pmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{pmatrix} \quad T(x) = Ax,$$

so  $T: \mathbf{R}^3 \rightarrow \mathbf{R}^2$ . Is  $T$  one-to-one? Is it onto?

The reduced row echelon form of  $A$  is

$$\begin{pmatrix} 1 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix},$$

which does not have a pivot in every row or in every column. Hence  $T$  is neither one-to-one nor onto.

[interactive]

## Summary

- ▶ A transformation  $T$  is **one-to-one** if  $T(x) = b$  has *at most one* solution, for every  $b$  in  $\mathbf{R}^m$ .
- ▶ A transformation  $T$  is **onto** if  $T(x) = b$  has *at least one* solution, for every  $b$  in  $\mathbf{R}^m$ .
- ▶ A matrix transformation with matrix  $A$  is one-to-one if and only if the columns of  $A$  are linearly independent, if and only if  $A$  has a pivot in every column.
- ▶ A matrix transformation with matrix  $A$  is onto if and only if the columns of  $A$  span  $\mathbf{R}^m$ , if and only if  $A$  has a pivot in every row.
- ▶ Two of the most basic questions one can ask about a transformation is whether it is one-to-one or onto.