

MATH 1553
SAMPLE FINAL EXAM, SPRING 2018

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Circle the name of your instructor below:

Fathi Jankowski Kordek Strenner Yan

Please **read all instructions** carefully before beginning.

- Each problem is worth 10 points. The maximum score on this exam is 100 points.
- You have 170 minutes to complete this exam.
- You may not use any calculators or aids of any kind (notes, text, etc.).
- Please show your work. A correct answer without appropriate work will receive little or no credit.
- You may cite any theorem proved in class or in the sections we covered in the text.
- Check your answers if you have time left! Most linear algebra computations can be easily verified for correctness.
- Good luck!

This is a practice exam. It is roughly similar in format, length, and difficulty to the real exam. It is not meant as a comprehensive list of study problems.

Problem 1.

True or false. Circle **T** if the statement is *always* true. Otherwise, circle **F**.

You do not need to justify your answer, and there is no partial credit.

In each case, assume that the entries of all matrices and all vectors are real numbers.

- a) **T** **F** If A is an $n \times n$ matrix and $\text{rank}(A) = 1$, then every column vector of A lies on the same line through the origin in \mathbf{R}^n .

- b) **T** **F** The transformation $T : \mathbf{R}^3 \rightarrow \mathbf{R}^3$ given below is linear.

$$T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x - y \\ x + y \\ z + 1 \end{pmatrix}.$$

- c) **T** **F** If $A = \begin{pmatrix} 2 & 5 \\ 0 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 4 & 10 \\ 0 & 6 \end{pmatrix}$, then A and B are similar.

- d) **T** **F** If A is a 3×3 matrix with three orthogonal eigenvectors, then A is diagonalizable.

- e) **T** **F** If $W = \text{Span}\{v_1, v_2\}$, then

$$\text{proj}_W(x) = \frac{x \cdot v_1}{v_1 \cdot v_1} v_1 + \frac{x \cdot v_2}{v_2 \cdot v_2} v_2.$$

Problem 2.

Short answer questions: you need not explain your answers, but show any computations in part (d). In each case, assume that the entries of all matrices are real numbers.

- a) Give an example of a 3×3 matrix whose eigenspace corresponding to the eigenvalue $\lambda = 4$ is a two-dimensional plane.

b) Let $A = \begin{pmatrix} a & 15 & 7 \\ 0 & 3 & 5 \\ 0 & 0 & \frac{1}{6} \end{pmatrix}$.

A is not invertible when $a = \underline{\hspace{2cm}}$.

In this case, A is / is not diagonalizable (circle one.)

- c) Suppose A is a 3×3 matrix. Which of the following are possible?
(Circle all that apply.)

- (1) All of its eigenvalues are real, and the matrix is not diagonalizable.
- (2) Its eigenspace corresponding to the eigenvalue $\lambda = -5$ is a plane, and the algebraic multiplicity of -5 as an eigenvalue is 1.
- (3) Every nonzero vector in \mathbf{R}^3 is an eigenvector of A .

- d) Find the area of the triangle with vertices $(-3, 1)$, $(0, 2)$, $(-1, -2)$.

Problem 3.

Short answer questions: you need not explain your answers. In each case, assume that the entries of all matrices and vectors are real numbers.

a) Which of the following are subspaces of \mathbf{R}^3 ? Circle all that apply.

(1) $\text{Nul}(A)$, where $A = \begin{pmatrix} 1 & 0 & 2 \\ -1 & 2 & 0 \\ 0 & 3 & 3 \\ 3 & 1 & 4 \end{pmatrix}$.

(2) The set of solutions to $T(v) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, where $T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ y \\ x \end{pmatrix}$.

(3) The eigenspace corresponding to $\lambda = 1$, for any 3×3 matrix B that has 1 as an eigenvalue.

b) Let $T : \mathbf{R}^4 \rightarrow \mathbf{R}^3$ be a linear transformation with standard matrix A , so $T(v) = Av$. Which of the following are possible? *Circle all that apply.*

(1) The equation $Ax = 0$ has only the trivial solution.

(2) $\text{rank}(A) = \dim(\text{Nul } A)$.

(3) The equation $Ax = b$ is consistent for each b in \mathbf{R}^3 .

c) Suppose $\det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = -2$. Find $\det(3A)$ if $A = \begin{pmatrix} -4a + d & -4b + e & -4c + f \\ a & b & c \\ g & h & i \end{pmatrix}$.

d) Let v, w in \mathbf{R}^6 be orthogonal vectors with $\|v\| = 2$ and $\|w\| = 3$. Let

$$x = 3v - w \quad y = v + w.$$

Find the dot product $x \cdot y$

Problem 4.

- a) Let $T : \mathbf{R}^2 \rightarrow \mathbf{R}^2$ be the rotation counterclockwise by 90 degrees. Find the standard matrix A for T (in other words, $T(v) = Av$).

- b) Let $U : \mathbf{R}^3 \rightarrow \mathbf{R}^2$ be the linear transformation given by

$$U \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z - x \\ x + y + z \end{pmatrix}.$$

Find the standard matrix B for U .

- c) Compute $(T \circ U) \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$.

Problem 5.

Consider the subspace V of \mathbf{R}^4 given by

$$V = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \mid x - 2y + 5z = 0 \text{ and } -\frac{z}{2} + w = 0 \right\}.$$

a) Find a basis for V .

b) Find a basis for V^\perp .

c) Is there a matrix A so that $\text{Col}(A) = V$? If so, find such an A . If not, justify why no such A exists.

Problem 7.

Let $A = \begin{pmatrix} -2 & 5 \\ -2 & 4 \end{pmatrix}$.

- a) Find the (complex) eigenvalues of A . For full credit, you must write your answers in the spaces below.

The eigenvalue with *positive* imaginary part is $\lambda_1 = \underline{\hspace{2cm}}$.

The eigenvalue with *negative* imaginary part is $\lambda_2 = \underline{\hspace{2cm}}$.

- b) For each of the eigenvalues of A , find an eigenvector.

For full credit, you must write your answers in the spaces below.

An eigenvector for λ_1 (the eigenvalue with *positive* imaginary part) is $v_1 = \begin{pmatrix} \\ \end{pmatrix}$.

An eigenvector for λ_2 (the eigenvalue with *negative* imaginary part) is $v_2 = \begin{pmatrix} \\ \end{pmatrix}$.

Problem 8.

A matrix A and its reduced row echelon form are given below.

$$A = \begin{pmatrix} 1 & -2 & -2 & -1 \\ 2 & -4 & 1 & 3 \\ 1 & -2 & 0 & 1 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & -2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

a) Find an orthogonal basis for $\text{Nul}(A)$.

b) Find an orthogonal basis for $\text{Col}(A)$.

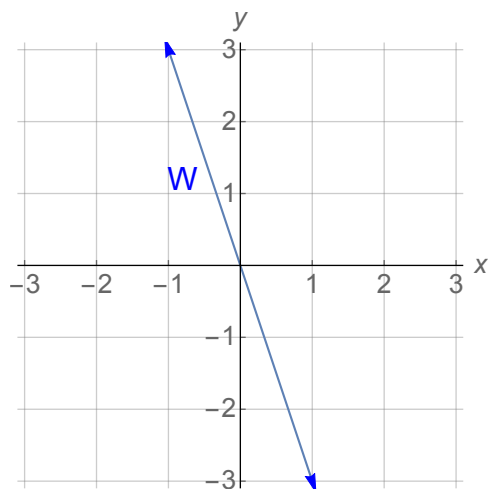
c) Find the vector in $\text{Col}(A)$ that is closest to $\begin{pmatrix} 0 \\ 15 \\ 0 \end{pmatrix}$.

Problem 9.

Let W be the line $y = -3x$ in \mathbf{R}^2 , and let $T : \mathbf{R}^2 \rightarrow \mathbf{R}^2$ be the linear transformation corresponding to orthogonal projection onto W .

a) Find the standard matrix A for T .

b) Draw W^\perp below. Be precise!



c) Let $z = \begin{pmatrix} -2 \\ -1 \end{pmatrix}$. Find vectors z_W in W and z_{W^\perp} in W^\perp so that $z = z_W + z_{W^\perp}$.

Problem 10.

Find the least-squares line $y = Mx + B$ that approximates the data points

$$(-2, -11), \quad (0, -2), \quad (4, 2).$$

Scratch paper. This sheet will not be graded under any circumstances.