MATH 1553, SPRING 2018
SAMPLE MIDTERM 1: THROUGH SECTION 1.5

Name		**Section**	

Please **read all instructions** carefully before beginning.

- You have 50 minutes to complete this exam.
- There are no aids of any kind (calculators, notes, text, etc.) allowed.
- Please show your work unless specified otherwise. A correct answer without appropriate work may be given little or no credit.
- You may cite any theorem proved in class or in the sections we covered in the text.
- Good luck!

This is a practice exam. It is similar in format, length, and difficulty to the real exam. It is **not** meant as a comprehensive list of study problems. I recommend completing the practice exam in 50 minutes, without notes or distractions.

Problem 1.

[Parts a) through e) are worth 2 points each]

True or false. Circle **T** if the statement is **always** true, and circle **F** otherwise. You do not need to justify your answer.

a) **T** **F** If $Ax = b$ is consistent, then the equation $Ax = 5b$ is consistent.

b) **T** **F** If a system of linear equations has more variables than equations, then the system must have infinitely many solutions.

c) **T** **F** If A is an $m \times n$ matrix and $Ax = 0$ has a unique solution, then $Ax = b$ is consistent for every b in $\mathbf{R}^m$.

d) **T** **F** The three vectors $\begin{pmatrix}1\\0\\0\end{pmatrix}$, $\begin{pmatrix}1\\1\\0\end{pmatrix}$, and $\begin{pmatrix}-1\\0\\1\end{pmatrix}$ span $\mathbf{R}^3$.

e) **T** **F** If A is a 5×3 matrix, then it is possible for $Ax = 0$ to be inconsistent.

Solution.

a) True. If $Aw = b$ then $A(5w) = 5Aw = 5b$.

b) False. The system can be inconsistent. For example: $x + y + z = 5, \quad x + y + z = 2$.

c) False. For example, if $A = \begin{pmatrix}1 & 0\\0 & 1\\0 & 0\end{pmatrix}$, then $Ax = 0$ has only the trivial solution, but

$$Ax = \begin{pmatrix}0\\0\\1\end{pmatrix}$$ has no solution.

d) True. The three vectors form a 3×3 matrix with a pivot in every row.

e) False. Every homogeneous system is consistent.

Problem 2.

Parts (a), (b) and (c) are 2 points each. Parts (d) and (e) are 3 points each.

a) Compute $\begin{pmatrix} 3 & 2 \\ -2 & 0 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ -3 \end{pmatrix}$

b) If A is a 2×3 matrix with 2 pivots, then the set of solutions to $Ax = 0$ is a:

(circle one answer) point line 2-plane 3-plane

in:

(circle one answer) $\mathbf{R}$ $\mathbf{R}^2$ $\mathbf{R}^3$.

c) Write a vector equation which represents an inconsistent system of two linear equations in x_1 and x_2.

d) Write a vector in $\mathbf{R}^3$ which is not a linear combination of $v_1 = \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 5 \\ 2 \\ 2 \end{pmatrix}$.

Justify your answer.

e) If b, v, w are vectors in $\mathbf{R}^3$ and $\text{Span}\{b, v, w\} = \mathbf{R}^3$, is it possible that b is in $\text{Span}\{v, w\}$? Justify your answer.

Solution.

a) $$1\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} - 3\begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} + \begin{pmatrix} -6 \\ 0 \\ -12 \end{pmatrix} = \begin{pmatrix} -3 \\ -2 \\ -11 \end{pmatrix}.$$

b) Line in $\mathbf{R}^3$. Since there are 2 pivots but 3 columns, one column will not have a pivot, so $Ax = 0$ will have exactly one free variable. The number of entries in x must match the number of columns of A (namely, 3), so each solution x is in $\mathbf{R}^3$.

c) The system $\left(\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 1 \end{array}\right)$ is inconsistent; its corresponding vector equation is

$$x_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} + x_2\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

d) If v is a linear combination of v_1 and v_2, then $v = cv_1 + dv_2 = \begin{pmatrix} 4c + 5d \\ c + 2d \\ c + 2d \end{pmatrix}$ for some scalars c and d, so the second entry of v must equal its third entry. Therefore, a vector such as $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ cannot be a linear combination of v_1 and v_2.

e) No. Recall that $\text{Span}\{b, v, w\}$ is the set of all linear combinations of b, v, and w. If b is in $\text{Span}\{v, w\}$ then b is a linear combination of v and w. Consequently, any element of $\text{Span}\{b, v, w\}$ is a linear combination of v and w and is therefore in $\text{Span}\{v, w\}$, which is at most a 2-plane and cannot be all of $\mathbf{R}^3$.

To see why the span of v and w can never be $\mathbf{R}^3$, consider the matrix A whose columns are v and w. Since A is 3×2, it has at most two pivots, so A cannot have a pivot in every row. Therefore, by a theorem from section 1.4, the equation $Ax = b$ will fail to be consistent for some b in $\mathbf{R}^3$, which means that some b in $\mathbf{R}^3$ is not in the span of v and w.

Problem 3. [10 points]

Johnny Rico believes that the secret to the universe can be found in the system of two linear equations in x and y given by

$$x - y = h$$
$$3x + hy = 4$$

where h is a real number.

a) Find all values of h (if any) which make the system inconsistent. Briefly justify your answer.

b) Find all values of h (if any) which make the system have a unique solution. Briefly justify your answer.

Solution.

Represent the system with an augmented matrix and row-reduce:

$$\left(\begin{array}{cc|c} 1 & -1 & h \\ 3 & h & 4 \end{array}\right) \xrightarrow{R_2 - 3R_1} \left(\begin{array}{cc|c} 1 & -1 & h \\ 0 & h+3 & 4-3h \end{array}\right).$$

a) If $h = -3$ then the matrix is $\left(\begin{array}{cc|c} 1 & -1 & -3 \\ 0 & 0 & 13 \end{array}\right)$, which has a pivot in the rightmost column and is therefore inconsistent.

b) If $h \neq -3$, then the matrix has a pivot in each row to the left of the augment: $\left(\begin{array}{cc|c} \boxed{1} & -1 & h \\ 0 & \boxed{\text{h+3}} & 4-3h \end{array}\right)$. The right column is not a pivot column, so the system is consistent. The left side has a pivot in each column, so the solution is unique.

Problem 4. [11 points]

a) Solve the system of equations by putting an augmented matrix into reduced row echelon form. Clearly indicate which variables (if any) are free variables.

$$\begin{aligned} x_1 + 2x_2 + 2x_3 - x_4 &= 4 \\ 2x_1 + 4x_2 + x_3 - 2x_4 &= -1 \\ -x_1 - 2x_2 - x_3 + x_4 &= -1 \end{aligned}$$

b) Write the set of solutions to

$$\begin{aligned} x_1 + 2x_2 + 2x_3 - x_4 &= 0 \\ 2x_1 + 4x_2 + x_3 - 2x_4 &= 0 \\ -x_1 - 2x_2 - x_3 + x_4 &= 0 \end{aligned}$$

in parametric vector form.

Solution.

a)

$$\left(\begin{array}{cccc|c} 1 & 2 & 2 & -1 & 4 \\ 2 & 4 & 1 & -2 & -1 \\ -1 & -2 & -1 & 1 & -1 \end{array}\right) \xrightarrow[R_3=R_3+R_1]{R_2=R_2-2R_1} \left(\begin{array}{cccc|c} 1 & 2 & 2 & -1 & 4 \\ 0 & 0 & -3 & 0 & -9 \\ 0 & 0 & 1 & 0 & 3 \end{array}\right) \xrightarrow{R_2\longleftrightarrow R_3} \left(\begin{array}{cccc|c} 1 & 2 & 2 & -1 & 4 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & -3 & 0 & -9 \end{array}\right)$$

$$\xrightarrow[R_1=R_1-2R_2]{R_3=R_3+3R_2} \left(\begin{array}{cccc|c} \boxed{1} & 2 & 0 & -1 & -2 \\ 0 & 0 & \boxed{1} & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Therefore, x_2 and x_4 are free, and we have:

$$\boxed{x_1 = -2 - 2x_2 + x_4 \qquad x_2 = x_2 \qquad x_3 = 3 \qquad x_4 = x_4}.$$

b) If we had written the solution to part (a) in parametric vector form, it would be:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -2 - 2x_2 + x_4 \\ x_2 \\ 3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ 3 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

The equation in (b) is just the corresponding homogeneous equation, which is a translate of the above plane which includes the origin.

$$\boxed{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \qquad (x_2, x_4 \text{ real})}.$$

Problem 5. [7 points]

Write an augmented matrix corresponding to a system of two linear equations in three variables x_1, x_2, x_3, whose solution set is the span of $\begin{pmatrix} -4 \\ 1 \\ 0 \end{pmatrix}$.

Briefly justify your answer.

Solution.

This problem is familiar territory, except that here, we are asked to come up with a system with the prescribed span, rather than being handed a system and discovering the span.

Since the span of any vector includes the origin, the zero vector is a solution, so the system is homogeneous.

Note that the span of $\begin{pmatrix} -4 \\ 1 \\ 0 \end{pmatrix}$ is all vectors of the form $t\begin{pmatrix} -4 \\ 1 \\ 0 \end{pmatrix}$ where t is real.

It consists of all $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ so that $x_1 = -4x_2,\ \ x_2 = x_2,\ \ x_3 = 0$.

The equation $x_1 = -4x_2$ gives $x_1 + 4x_2 = 0$, so one line in the matrix can be $\left(\begin{array}{ccc|c} 1 & 4 & 0 & 0 \end{array}\right)$. The equation $x_3 = 0$ translates to $\left(\begin{array}{ccc|c} 0 & 0 & 1 & 0 \end{array}\right)$. Note that this leaves x_2 free, as desired.

This gives us the augmented matrix

$$\boxed{\left(\begin{array}{ccc|c} 1 & 4 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)}.$$

(Multiple examples are possible)

Let's check: the system has one free variable x_2.
The first line says $x_1 + 4x_2 = 0$, so $x_1 = -4x_2$. The second line says $x_3 = 0$.
Therefore, the general solution is $x = \begin{pmatrix} -4x_2 \\ x_2 \\ 0 \end{pmatrix} = x_2\begin{pmatrix} -4 \\ 1 \\ 0 \end{pmatrix}$ where x_2 is real.

In other words, the solution set is the span of $\begin{pmatrix} -4 \\ 1 \\ 0 \end{pmatrix}$.

The system of equations is

$$\begin{aligned} x_1 + 4x_2 &= 0 \\ x_3 &= 0. \end{aligned}$$

[Scratch work]