Math 1553 Worksheet §§6.1–6.5

Solutions

1. **a)** True or false: If u, v, w are vectors in $\mathbf{R}^n$ with $u \perp v$ and $v \perp w$, then $u \perp w$.

b) Find the standard matrix B for proj_L, where $L = \text{Span}\left\{\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}\right\}$.

Solution.

a) False. For example, take $u = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, and $w = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Then $u \perp v$ and $v \perp w$ but $u \cdot w = 1$.

b) The columns of B are $\text{proj}_L(e_1)$, $\text{proj}_L(e_2)$, and $\text{proj}_L(e_3)$. Letting $u = (1, 1, -1)$, we compute

$$\text{proj}_L(e_1) = \frac{e_1 \cdot u}{u \cdot u} u = \frac{1}{3}\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

$$\text{proj}_L(e_2) = \frac{e_2 \cdot u}{u \cdot u} u = \frac{1}{3}\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

$$\text{proj}_L(e_3) = \frac{e_3 \cdot u}{u \cdot u} u = -\frac{1}{3}\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

$$\implies B = \frac{1}{3}\begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}.$$

2. Find an orthogonal basis for the subspace of $\mathbf{R}^4$ spanned by $\begin{pmatrix}1\\-1\\1\\1\end{pmatrix}$, $\begin{pmatrix}6\\-2\\2\\6\end{pmatrix}$, and $\begin{pmatrix}4\\20\\-14\\10\end{pmatrix}$.

Solution.

Let $v_1 = \begin{pmatrix}1\\-1\\1\\1\end{pmatrix}, \quad v_2 = \begin{pmatrix}6\\-2\\2\\6\end{pmatrix}, \quad v_3 = \begin{pmatrix}4\\20\\-14\\10\end{pmatrix}$.

We apply Gram–Schmidt to $\{v_1, v_2, v_3\}$ to get an orthogonal basis $\{u_1, u_2, u_3\}$.

$u_1 = v_1$

$$u_2 = v_2 - \frac{v_2 \cdot u_1}{u_1 \cdot u_1} u_1 = \begin{pmatrix}6\\-2\\2\\6\end{pmatrix} - \frac{16}{4}\begin{pmatrix}1\\-1\\1\\1\end{pmatrix} = \begin{pmatrix}2\\2\\-2\\2\end{pmatrix}$$

$$u_3 = v_3 - \frac{v_3 \cdot u_1}{u_1 \cdot u_1} u_1 - \frac{v_3 \cdot u_2}{u_2 \cdot u_2} u_2 = \begin{pmatrix}4\\20\\-14\\10\end{pmatrix} + \frac{20}{4}\begin{pmatrix}1\\-1\\1\\1\end{pmatrix} - \frac{96}{16}\begin{pmatrix}2\\2\\-2\\2\end{pmatrix} = \begin{pmatrix}-3\\3\\3\\3\end{pmatrix}.$$

3. Find the best fit line $y = Ax + B$ through the points $(0,0)$, $(1,8)$, $(3,8)$, and $(4,20)$.

Solution.

We want to find a least squares solution to the system of linear equations

$$\begin{aligned} 0 &= A(0) + B \\ 8 &= A(1) + B \\ 8 &= A(3) + B \\ 20 &= A(4) + B \end{aligned} \qquad \Longleftrightarrow \qquad \begin{pmatrix}0 & 1\\1 & 1\\3 & 1\\4 & 1\end{pmatrix}\begin{pmatrix}A\\B\end{pmatrix} = \begin{pmatrix}0\\8\\8\\20\end{pmatrix}.$$

We compute

$$\begin{pmatrix}0 & 1 & 3 & 4\\1 & 1 & 1 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\1 & 1\\3 & 1\\4 & 1\end{pmatrix} = \begin{pmatrix}26 & 8\\8 & 4\end{pmatrix}$$

$$\begin{pmatrix}0 & 1 & 3 & 4\\1 & 1 & 1 & 1\end{pmatrix}\begin{pmatrix}0\\8\\8\\20\end{pmatrix} = \begin{pmatrix}112\\36\end{pmatrix}$$

$$\left(\begin{array}{cc|c}26 & 8 & 112\\8 & 4 & 36\end{array}\right) \xrightarrow{\text{rref}} \left(\begin{array}{cc|c}1 & 0 & 4\\0 & 1 & 1\end{array}\right).$$

Hence the least squares solution is $A = 4$ and $B = 1$, so the best fit line is $y = 4x + 1$.

Here is a picture with the best-fit line and (just for fun) the best-fit parabola for the data. The "best fit cubic" would be the cubic $y = \frac{5}{3}x^3 - \frac{28}{3}x^2 + \frac{47}{3}x$, which actually passes through all four points.