Math 1553 Worksheet §5.5

1. Answer true or false, and justify your answer. In each case, A is a matrix whose entries are real.

a) If A is the matrix that implements rotation by 143° in $\mathbf{R}^2$, then A has no real eigenvalues.

b) A 3×3 matrix can have a non-real complex eigenvalue with multiplicity 2.

c) A 3×3 matrix can have eigenvalues $3, 5$, and $2+i$.

Solution.

a) True. If A had a real eigenvalue λ, then we would have $Ax = \lambda x$ for some vector x in $\mathbf{R}^2$. This means that x would lie on the same line through the origin as the rotation of x by 143°, which is impossible.

b) False. If c is a (non-real) complex eigenvalue with multiplicity 2, then its conjugate $\bar{c}$ is an eigenvalue with multiplicity 2 since complex eigenvalues always occur in conjugate pairs. This would mean A has a characteristic polynomial of degree 4 or more, which is impossible for a 3×3 matrix.

c) False. If $2+i$ is an eigenvalue then so is its conjugate $2-i$.

2. Let $A = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$.

a) Find all eigenvalues and eigenvectors of A.

b) Using the eigenvalue with negative imaginary part, write $A = PCP^{-1}$, where C is a rotation followed by a scale. Describe what A does geometrically.

Solution.

a) The characteristic polynomial is

$$\lambda^2 - \mathrm{Tr}(A)\lambda + \det(A) = \lambda^2 - 2\lambda + 5$$

$$\lambda^2 - 2\lambda + 5 = 0 \iff \lambda = \frac{2 \pm \sqrt{4-20}}{2} = \frac{2 \pm 4i}{2} = 1 \pm 2i.$$

For the eigenvalue $\lambda = 1-2i$, we use the trick from class: the first row $\begin{pmatrix} a & b \end{pmatrix}$ of $A-\lambda I$ will lead to an eigenvector $\begin{pmatrix} -b \\ a \end{pmatrix}$ (or equivalently, $\begin{pmatrix} b \\ -a \end{pmatrix}$ if you prefer).

$$\left(A-(1-2i)I \mid 0\right) = \left(\begin{array}{cc|c} 2i & 2 & 0 \\ (*) & (*) & 0 \end{array}\right) \implies v = \begin{pmatrix} -2 \\ 2i \end{pmatrix}.$$

From the correspondence between conjugate eigenvalues and their eigenvectors, we know (without doing any additional work!) that for the eigenvalue $\lambda = 1+2i$, a corresponding eigenvector is $w = \overline{v} = \begin{pmatrix} -2 \\ -2i \end{pmatrix}$.

b) We use $\lambda = 1-2i$ and its associated $v = \begin{pmatrix} -2 \\ 2i \end{pmatrix}$.

$A = PCP^{-1}$ where $P = \begin{pmatrix} \mathrm{Re}(v) & \mathrm{Im}(v) \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}$ and

$$C = \begin{pmatrix} \mathrm{Re}(\lambda) & \mathrm{Im}(\lambda) \\ -\mathrm{Im}(\lambda) & \mathrm{Re}(\lambda) \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}.$$

The scale is by a factor of $|\lambda| = |1+2i| = \sqrt{1^2+2^2} = \sqrt{5}$. If we factor this out of C we get

$$C = \sqrt{5}\begin{pmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}.$$

We can use our known formula for rotation matrices (Ch. 1) to find the angle of rotation, or alternatively we can plug in the formula we saw in class in 5.5.

Rotation matrix formula: Let θ be the angle of counterclockwise rotation. From the first row we see $\cos(\theta) = \frac{1}{\sqrt{5}}$ and $-\sin(\theta) = -\frac{2}{\sqrt{5}}$, so $\sin(\theta) = \frac{2}{\sqrt{5}}$. Therefore, θ is in the first quadrant and $\tan(\theta) = 2$, hence $\theta = \arctan(2)$.

C is counterclockwise rotation by the angle $\arctan(2)$, followed by scaling by a factor of $\sqrt{5}$.

Alternative for finding angle (from section 5.5): Rotation is counterclockwise by $\arg(\overline{\lambda})$ (or equivalently, by $-\arg(\lambda)$). Since $\lambda = 1-2i$, we have $\overline{\lambda} = 1+2i$, and $\arg(\overline{\lambda})$ lies in quadrant 1 and has tangent $\frac{2}{1}$, hence $\arg(\overline{\lambda}) = \arctan(2)$.

See the [interactive] demo for how A acts geometrically.

***Note: there are multiple answers possible for part **b)**.

For example, for the eigenvector we could have used $\begin{pmatrix} b \\ -a \end{pmatrix}$ where $\begin{pmatrix} a & b \end{pmatrix}$ is the first row of $A - \lambda I$.

Row 1 of $A - \lambda I$ was $\begin{pmatrix} 2i & 2 \end{pmatrix}$, so $\begin{pmatrix} 2 \\ -2i \end{pmatrix}$ as an eigenvector.

This would give us $P = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$ rather than $P = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}$. However, it would still be the case that $A = PCP^{-1}$ since

$$PCP^{-1} = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} = A.$$