Math 1553 Worksheet §2.8, 2.9

1. Let $A = \begin{pmatrix} 1 & 3 & 2 \\ 2 & 6 & 4 \end{pmatrix}$ and let T_A be the corresponding linear transformation, defined by $T_A(v) = Av$.

a) Find a basis for the nullspace of A and roughly sketch Nul A.

b) Using the previous part, determine if T_A is one-to-one.

c) Find a basis for the column space of A and draw a picture of Col A.

d) Using the previous part, determine if T_A is onto.

e) Given that we know the nullspace and column space, we can describe the transformation T_A as follows:

The transformation T_A squashes $\mathbf{R}^{\square}$ to a $\boxed{}$ by crushing parallel $\boxed{}$ to points. Its range is a $\boxed{}$ in $\mathbf{R}^{\square}$.

Solution.

a) We need to find the parametric vector form for the solutions of $Ax = 0$. After row reducing, we get the matrix $\begin{pmatrix} 1 & 3 & 2 \\ 0 & 0 & 0 \end{pmatrix}$. So we have $x_1 = -3x_2 - 2x_3$, $x_2 = x_2$, $x_3 = x_3$ which gives

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}.$$

So a basis for the nullspace is $\left\{ \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \right\}$.

The picture is a plane is $\mathbf{R}^3$.

b) T_A is not one-to-one, because the nullspace is bigger than just the zero vector.

c) Only the first column of A is a pivot column, so a basis for the columns space is the first column of A: $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

The picture is the line $y = 2x$ in $\mathbf{R}^2$.

d) T_A is not onto, since the columns space is only a line in $\mathbf{R}^2$, and not all of $\mathbf{R}^2$.

e) The transformation T_A squashes $\mathbf{R}^{\boxed{3}}$ [domain] to a $\boxed{\text{line}}$ [column space] by crushing parallel $\boxed{\text{planes}}$ [nullspace] to points. Its range is a $\boxed{\text{line}}$ [column space] in $\mathbf{R}^{\boxed{2}}$ [codomain].

2. Answer "YES" if the statement is always true, "NO" if it is always false, and "MAYBE" otherwise.

a) If A is a 3×100 matrix of rank 2, then $\dim(\mathrm{Nul}A) = 97$.

YES NO MAYBE

b) If A is an $m \times n$ matrix and $Ax = 0$ has only the trivial solution, then the columns of A form a basis for $\mathbf{R}^m$.

YES NO MAYBE

Solution.

a) No. By the Rank Theorem, $\mathrm{rank}(A) + \dim(\mathrm{Nul}A) = 100$, so $\dim(\mathrm{Nul}A) = 98$.

b) Maybe. If $Ax = 0$ has only the trivial solution and $m = n$, then A is invertible, so the columns of A are linearly independent and span $\mathbf{R}^m$.

If $m > n$ then the statement is false. For example, $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$ has only the trivial solution for $Ax = 0$, but its column space is a 2-dimensional subspace of $\mathbf{R}^3$.

3. Consider the following vectors in $\mathbf{R}^3$:

$$b_1 = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} \qquad b_2 = \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix} \qquad u = \begin{pmatrix} 1 \\ 10 \\ 7 \end{pmatrix}$$

Let $V = \mathrm{Span}\{b_1, b_2\}$.

a) Explain why $\mathcal{B} = \{b_1, b_2\}$ is a basis for V.

b) Determine if u is in V. If so, find $[u]_{\mathcal{B}}$ (the $\mathcal{B}$-coordinates of u).

Solution.

a) A quick check shows that b_1 and b_2 are linearly independent, and we already know they span V, so $\{b_1, b_2\}$ is a basis for V.

b) u is in V if and only if $c_1 b_1 + c_2 b_2 = u$ for some c_1 and c_2, in which case $[u]_{\mathcal{B}} = \binom{c_1}{c_2}$. We form the augmented matrix $\left(b_1 \quad b_2 \mid u \right)$ and solve:

$$\left(\begin{array}{cc|c} 2 & 1 & 1 \\ 2 & 4 & 10 \\ 2 & 3 & 7 \end{array}\right) \xrightarrow[R_3 = R_3 - R_1]{R_2 = R_2 - R_1} \left(\begin{array}{cc|c} 2 & 1 & 1 \\ 0 & 3 & 9 \\ 0 & 2 & 6 \end{array}\right) \xrightarrow[R_2 = R_2/3]{R_3 = R_3 - \frac{2}{3}R_2} \left(\begin{array}{cc|c} 2 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right) \xrightarrow[\text{then } R_1 = R_1/2]{R_1 = R_1 - R_2} \left(\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right).$$

We found $c_1 = -1$ and $c_2 = 3$. This means $-b_1 + 3b_2 = u$, so u is in $\mathrm{Span}\{b_1, b_2\}$ and $[u]_{\mathcal{B}} = \binom{-1}{3}$.