Math 1553 Worksheet §2.1, 2.2, 2.3

Solutions

1. If A is a 3×5 matrix and B is a 3×2 matrix, which of the following are defined?

a) $A-B$

b) AB

c) A^TB

d) A^2

Solution.

Only (c).
$A-B$ is nonsense. In order for $A-B$ to be defined, A and B need to have the same number of rows and same number of columns as each other.
AB is undefined since the number of columns of A does not equal the number of rows of B.
A^T is 5×3 and B is 3×2, so A^TB is a 5×2 matrix.
A^2 is nonsense (can't do 3×5 times 3×5).

2. True or false (justify your answer). Answer true if the statement is *always* true. Otherwise, answer false.

a) If A is an $n\times n$ matrix and the equation $Ax=b$ has at least one solution for each b in $\mathbf{R}^n$, then the solution is *unique* for each b in $\mathbf{R}^n$.

b) If A is an $n\times n$ matrix and every vector in $\mathbf{R}^n$ can be written as a linear combination of the columns of A, then A is invertible.

c) If A and B are invertible $n\times n$ matrices, then $A+B$ is invertible and

$$(A+B)^{-1}=A^{-1}+B^{-1}.$$

d) If A is an $m\times n$ matrix and B is an $n\times p$ matrix, then each column of AB is a linear combination of the columns of A.

e) If $AB=BC$ and B is invertible, then $A=C$.

Solution.

a) True. The first part says the transformation $T:\mathbf{R}^n\to\mathbf{R}^n$ given by $T(x)=Ax$ is onto. Since A is $n\times n$, this is the same as saying A is invertible, so T is one-to-one and onto. Therefore, the equation $Ax=b$ has exactly one solution for each b in $\mathbf{R}^n$.

b) True. If the columns of A span $\mathbf{R}^n$, then A is invertible by the Invertible Matrix Theorem. We can also see this directly without quoting the IMT:

If the columns of A span $\mathbf{R}^n$, then A has n pivots, so A has a pivot in each row and column, hence its associated transformation $T(x) = Ax$ is one-to-one and onto and thus invertible. Therefore, A is invertible.

c) False. $A+B$ might not be invertible in the first place. For example, if $A = I_2$ and $B = -I_2$ then $A+B = 0$ which is not invertible. Even in the case when $A+B$ is invertible, it still might not be true that $(A+B)^{-1} = A^{-1} + B^{-1}$. For example, $(I_2 + I_2)^{-1} = (2I_2)^{-1} = \frac{1}{2}I_2$, whereas $(I_2)^{-1} + (I_2)^{-1} = I_2 + I_2 = 2I_2$.

d) True. If we let $v_1, \ldots, v_p$ be the columns of B, then $AB = \begin{pmatrix} Av_1 & Av_2 & \cdots & Av_p \end{pmatrix}$, where Av_i is in the column span of A for every i (this is part of the definition of matrix multiplication of vectors).

e) False. It is not easy to come up with an example that shows it is false, but algebraically, we see that we need to multiply by B^{-1} on the right side of each equation to cancel the B in the equation AB:

$$AB = BC \qquad AB(B^{-1}) = BC(B^{-1}) \qquad AI_n = BCB^{-1} \qquad \boxed{A = BCB^{-1}}.$$

Here is an example demonstrating that false is the correct answer.

If $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and $C = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, then B is invertible (in fact $B = B^{-1}$) and $A \neq C$ but

$$AB = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad BC = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

3. Suppose A is an invertible 3×3 matrix and

$$A^{-1}e_1 = \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix}, \quad A^{-1}e_2 = \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}, \quad A^{-1}e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Find A.

Solution.

The columns of A^{-1} are

$$(A^{-1}e_1 \;\; A^{-1}e_2 \;\; A^{-1}e_3), \qquad \text{so} \qquad A^{-1} = \begin{pmatrix} 4 & 3 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

A is the inverse of A^{-1}, so we use the method from 2.2 to find $(A^{-1})^{-1}$. Row-reducing $(A^{-1} \mid I)$ eventually gives us

$$\left(\begin{array}{ccc|ccc} 4 & 3 & 0 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{\text{RREF}} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{2}{5} & -\frac{3}{5} & 0 \\ 0 & 1 & 0 & -\frac{1}{5} & \frac{4}{5} & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right),$$

so

$$A = \begin{pmatrix} \dfrac{2}{5} & -\dfrac{3}{5} & 0 \\ -\dfrac{1}{5} & \dfrac{4}{5} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$