**Math 1553 Supplement §2.1, 2.2, 2.3**

Solutions

**1.** Find all matrices $B$ that satisfy

$$\begin{pmatrix} 1 & -3 \\ -3 & 5 \end{pmatrix} B = \begin{pmatrix} -3 & -11 \\ 1 & 17 \end{pmatrix}.$$

## Solution.

$B$ must have two rows and two columns for the above to compute, so $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We calculate

$$\begin{pmatrix} 1 & -3 \\ -3 & 5 \end{pmatrix} B = \begin{bmatrix} a-3c & b-3d \\ -3a+5c & -3b+5d \end{bmatrix}.$$

Setting this equal to $\begin{pmatrix} -3 & -11 \\ 1 & 17 \end{pmatrix}$ gives us

$$a - 3c = -3,$$

$$-3a + 5c = 1,$$

$$\text{(solving gives us } a = 3,\ c = 2)$$

$$b - 3d = -11,$$

$$-3b + 5d = 17.$$

$$\text{(solving gives us } b = 1,\ d = 4).$$

Therefore, $B = \begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix}$.

**2.**

**a)** Fill in: $A$ and $B$ are invertible $n \times n$ matrices, then the inverse of $AB$ is ________.

**b)** If the columns of an $n \times n$ matrix $Z$ are linearly independent, is $Z$ necessarily invertible? Justify your answer.

**c)** If $A$ and $B$ are $n \times n$ matrices and $ABx = 0$ has a unique solution, does $Ax = 0$ necessarily have a unique solution? Justify your answer.

## Solution.

**a)** $(AB)^{-1} = B^{-1}A^{-1}$.

**b)** Yes. The transformation $x \to Zx$ is one-to-one since the columns of $Z$ are linearly independent. Thus $Z$ has a pivot in all $n$ columns, so $Z$ has $n$ pivots. Since $Z$ also has $n$ rows, this means that $Z$ has a pivot in every row, so $x \to Zx$ is onto. Therefore, $Z$ is invertible.

Alternatively, since $Z$ is an $n \times n$ matrix whose columns are linearly independent, the Invertible Matrix Theorem (2.3) in 2.3 says that $Z$ is invertible.

**c)** Yes. Since $AB$ is an $n \times n$ matrix and $ABx = 0$ has a unique solution, the Invertible Matrix Theorem says that $AB$ is invertible. Note $A$ is invertible and its inverse is $B(AB)^{-1}$, since these are square matrices and

$$A(B(AB)^{-1}) = AB(AB)^{-1} = I_n.$$

Since $A$ is invertible, $Ax = 0$ has a unique solution by the Invertible Matrix Theorem.