Section 5.5

## Complex Eigenvalues

A Matrix with No Eigenvectors

Consider the matrix for the linear transformation for rotation by $\pi/4$ in the plane. The matrix is:

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

This matrix has no eigenvectors, as you can see geometrically: [interactive]

A

no nonzero vector x
is collinear with Ax
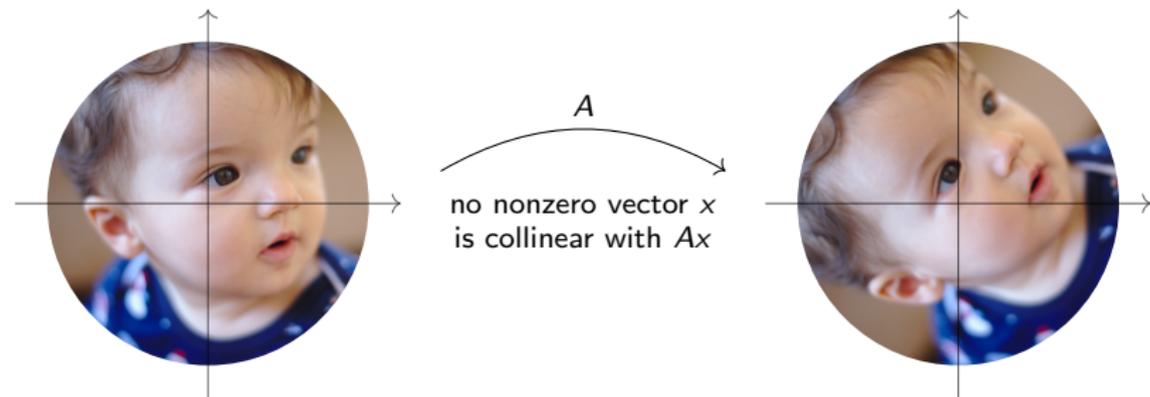
or algebraically:

$$f(\lambda) = \lambda^2 - \mathrm{Tr}(A)\,\lambda + \det(A) = \lambda^2 - \sqrt{2}\,\lambda + 1 \implies \lambda = \frac{\sqrt{2} \pm \sqrt{-2}}{2}.$$

It makes us sad that -1 has no square root. If it did, then $\sqrt{-2} = \sqrt{2} \cdot \sqrt{-1}$.

Mathematician's solution: we're just not using enough numbers! We're going to declare by *fiat* that there exists a square root of -1.

Definition
The number i is defined such that $i^2 = -1$.

Once we have i, we have to allow numbers like $a + bi$ for real numbers a, b.

Definition
A *complex number* is a number of the form $a + bi$ for a, b in **R**. The set of all complex numbers is denoted **C**.

Note **R** is contained in **C**: they're the numbers $a + 0i$.

We can identify **C** with $\mathbf{R}^2$ by $a + bi \longleftrightarrow \binom{a}{b}$. So when we draw a picture of **C**, we draw the plane:
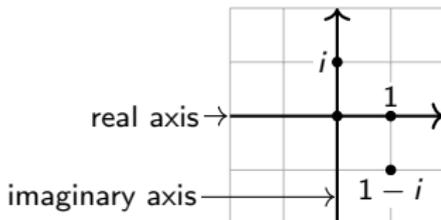
In the beginning, people only used counting numbers for, well, counting things: $1, 2, 3, 4, 5, \ldots$. Then someone (Persian mathematician Muḥammad ibn Mūsā al-Khwārizmī, 825) had the ridiculous idea that there should be a number 0 that represents an absence of quantity. This blew everyone's mind.

Then it occurred to someone (Chinese mathematician Liu Hui, c. 3rd century) that there should be *negative* numbers to represent a deficit in quantity. That seemed reasonable, until people realized that $10 - (-3)$ would have to equal 13. This is when people started saying, "bah, math is just too hard for me."

At this point it was inconvenient that you couldn't divide 2 by 3. Thus someone (Indian mathematician Aryabhatta, c. 5th century) invented fractions (rational numbers) to represent fractional quantities. These proved very popular. The Pythagoreans developed a whole belief system around the notion that any quantity worth considering could be broken down into whole numbers in this way.

Then the Pythagoreans (c. 6th century BCE) discovered that the hypotenuse of an isosceles right triangle with side length 1 (i.e. $\sqrt{2}$) is not a fraction. This caused a serious existential crisis and led to at least one death by drowning. The real number $\sqrt{2}$ was thus invented to solve the equation $x^2 - 2 = 0$.

So what's so strange about inventing a number i to solve the equation $x^2 + 1 = 0$? Is this really any stranger than saying an infinite nonrepeating decimal expansion represents a number?

Addition: $(2 - 3i) + (-1 + i) = 1 - 2i$.

Multiplication: $(2 - 3i)(-1 + i) = 2(-1) + 2i + 3i - 3i^2 = -2 + 5i + 3 = 1 + 5i$.

Complex conjugation: $\overline{a + bi} = a - bi$ is the **complex conjugate** of $a + bi$.
Check: $\overline{z + w} = \bar{z} + \bar{w}$ and $\overline{zw} = \bar{z} \cdot \bar{w}$.

Absolute value: $|a + bi| = \sqrt{a^2 + b^2}$. This is a *real* number.
Note: $(a + bi)(\overline{a + bi}) = (a + bi)(a - bi) = a^2 - (bi)^2 = a^2 + b^2$. So $|z| = \sqrt{z\bar{z}}$.
Check: $|zw| = |z| \cdot |w|$.

Division by a nonzero real number: $\dfrac{a + bi}{c} = \dfrac{a}{c} + \dfrac{b}{c}i$.

Division by a nonzero complex number: $\dfrac{z}{w} = \dfrac{z\bar{w}}{w\bar{w}} = \dfrac{z\bar{w}}{|w|^2}$.

Example:
$$\frac{1 + i}{1 - i} = \frac{(1 + i)^2}{1^2 + (-1)^2} = \frac{1 + 2i + i^2}{2} = i.$$

Real and imaginary part: $\operatorname{Re}(a + bi) = a \qquad \operatorname{Im}(a + bi) = b$.

Polar Coordinates for Complex Numbers

Any complex number $z = a + bi$ has the polar coordinates

$$z = |z|(\cos\theta + i\sin\theta).$$

The angle θ is called the **argument** of z, and is denoted $\theta = \arg(z)$. Note $\arg(\bar{z}) = -\arg(z)$.
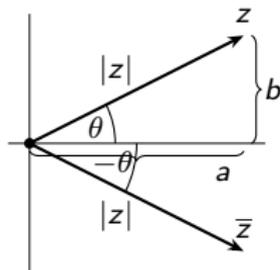
When you multiply complex numbers, you multiply the absolute values and add the arguments:

$$|zw| = |z|\,|w| \qquad \arg(zw) = \arg(z) + \arg(w).$$

The whole point of using complex numbers is to solve polynomial equations. It turns out that they are enough to find all solutions of all polynomial equations:
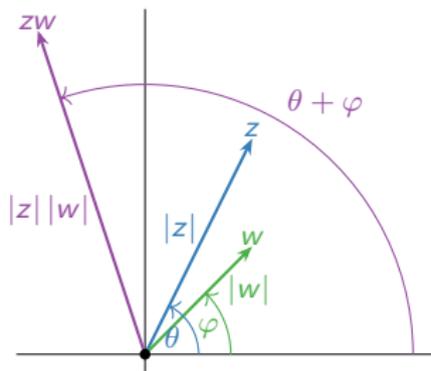
Fundamental Theorem of Algebra

Every polynomial of degree n has exactly n complex roots, counted with multiplicity.

Equivalently, if $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ is a polynomial of degree n, then

$$f(x) = (x - \lambda_1)(x - \lambda_2) \cdots (x - \lambda_n)$$

for (not necessarily distinct) complex numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$.

Important

If f is a polynomial with *real* coefficients, and if λ is a complex root of f, then so is $\overline{\lambda}$:

$$
\begin{aligned}
0 = \overline{f(\lambda)} &= \overline{\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0} \\
&= \overline{\lambda}^n + a_{n-1}\overline{\lambda}^{n-1} + \cdots + a_1\overline{\lambda} + a_0 = f(\overline{\lambda}).
\end{aligned}
$$

Therefore complex roots of real polynomials come in *conjugate pairs*.

Degree 2: The quadratic formula gives you the (real or complex) roots of any degree-2 polynomial:

$$f(x) = x^2 + bx + c \implies x = \frac{-b \pm \sqrt{b^2 - 4c}}{2}.$$

For instance, if $f(\lambda) = \lambda^2 - \sqrt{2}\lambda + 1$ then

$$\lambda = \frac{\sqrt{2} \pm \sqrt{-2}}{2} = \frac{\sqrt{2}}{2}(1 \pm i) = \frac{1 \pm i}{\sqrt{2}}.$$

Note the roots are complex conjugates if b, c are real.

Degree 3: A real cubic polynomial has either three real roots, or one real root and a conjugate pair of complex roots. The graph looks like:

or
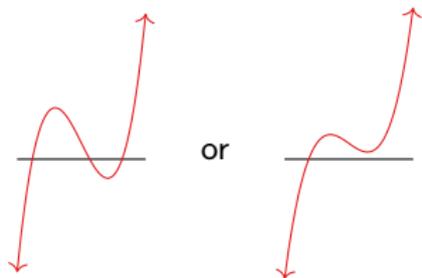
respectively. How do you find a real root? Sometimes you can use this:

Rational Root Theorem

Let f be a polynomial with integer coefficients:

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0.$$

Suppose that $a_0 \neq 0$ and $a_n \neq 0$. If p/q is a rational root (written in lowest terms), then

- p divides a_0, and
- q divides a_n.

Example: Factor $f(\lambda) = 5\lambda^3 - 18\lambda^2 + 21\lambda - 10$.

We guess that the polynomial has a rational root. By the rational root theorem, the possible numerators of the root are the divisors of 10, i.e., $\pm 1, \pm 2, \pm 5, \pm 10$, and the possible denominators are ± 1 and ± 5. Hence the root must be one of

$$\pm 1, \ \pm 2, \ \pm 5, \ \pm 10, \ \pm \frac{1}{5}, \ \pm \frac{2}{5}.$$

After trying some of these values for λ, we find $f(2) = 0$.

Since $f(2) = 0$, we can do polynomial long division by $\lambda - 2$: we get $f(\lambda) = (\lambda - 2)\left(5\lambda^2 - 8\lambda + 5\right)$. Using the quadratic formula, the second polynomial has a root when

$$\lambda = \frac{8 \pm \sqrt{64 - 100}}{10} = \frac{4}{5} \pm \frac{\sqrt{-36}}{10} = \frac{4 \pm 3i}{5}.$$

Therefore,

$$f(\lambda) = 5(\lambda - 2)\left(\lambda - \frac{4 + 3i}{5}\right)\left(\lambda - \frac{4 - 3i}{5}\right).$$

There is one real root, and a conjugate pair of complex roots.

The characteristic polynomial of

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

is $f(\lambda) = \lambda^2 - \sqrt{2}\lambda + 1$. This has two complex roots $(1 \pm i)/\sqrt{2}$.

Poll

Does A have any eigen*vectors*? If so, what are they?

Every matrix is guaranteed to have *complex* eigenvalues and eigenvectors. Using rotation by $\pi/4$ from before:

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \quad \text{has eigenvalues} \quad \lambda = \frac{1 \pm i}{\sqrt{2}}.$$

Let's compute an eigenvector for $\lambda = (1+i)/\sqrt{2}$:

$$A - \lambda I = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 - (1+i) & -1 \\ 1 & 1 - (1+i) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix}.$$

The second row is i times the first, so we row reduce:

$$\frac{1}{\sqrt{2}} \begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} \rightsquigarrow \frac{1}{\sqrt{2}} \begin{pmatrix} -i & -1 \\ 0 & 0 \end{pmatrix} \xrightarrow{\text{divide by } -i/\sqrt{2}} \begin{pmatrix} 1 & -i \\ 0 & 0 \end{pmatrix}.$$

The parametric form is $x = iy$, so an eigenvector is $\begin{pmatrix} i \\ 1 \end{pmatrix}$.

A similar computation shows that an eigenvector for $\lambda = (1-i)/\sqrt{2}$ is $\begin{pmatrix} -i \\ 1 \end{pmatrix}$.

So is $i \begin{pmatrix} -i \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ i \end{pmatrix}$ (you can scale by *complex* numbers).

Let A be a 2×2 matrix, and let λ be an eigenvalue of A.

Then $A - \lambda I$ is not invertible, so the second row is *automatically* a multiple of the first. (Think about it for a while: otherwise the rref is $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right)$.)

Hence the second row disappears in the rref, so *we don't care what it is!*

If $A - \lambda I = \begin{pmatrix} a & b \\ \star & \star \end{pmatrix}$, then $(A - \lambda I) \begin{pmatrix} -b \\ a \end{pmatrix} = 0$, so $\begin{pmatrix} -b \\ a \end{pmatrix}$ is an eigenvector.

So is $\begin{pmatrix} b \\ -a \end{pmatrix}$. (What if $a = b = 0$?)

Example:

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \qquad \lambda = \frac{1 - i}{\sqrt{2}}.$$

Then:

$$A - \lambda I = \frac{1}{\sqrt{2}} \begin{pmatrix} i & -1 \\ \star & \star \end{pmatrix}$$

so an eigenvector is

$$v = \begin{pmatrix} 1 \\ i \end{pmatrix}.$$

Conjugate Eigenvectors

For $A = \dfrac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$,

the eigenvalue $\dfrac{1+i}{\sqrt{2}}$ has eigenvector $\begin{pmatrix} 1 \\ -i \end{pmatrix}$.

the eigenvalue $\dfrac{1-i}{\sqrt{2}}$ has eigenvector $\begin{pmatrix} 1 \\ i \end{pmatrix}$.

Do you notice a pattern?

Fact

Let A be a real square matrix. If λ is a complex eigenvalue with eigenvector v, then $\overline{\lambda}$ is an eigenvalue with eigenvector $\overline{v}$.

Why?

$$Av = \lambda \implies A\overline{v} = \overline{Av} = \overline{\lambda v} = \overline{\lambda}\,\overline{v}.$$

> Both eigenvalues and eigenvectors of real square matrices occur in conjugate pairs.

A 3 × 3 Example

Find the eigenvalues and eigenvectors of

$$A = \begin{pmatrix} \frac{4}{5} & -\frac{3}{5} & 0 \\ \frac{3}{5} & \frac{4}{5} & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

The characteristic polynomial is

This factors out automatically if you expand cofactors along the third row or column

$$f(\lambda) = \det \begin{pmatrix} \frac{4}{5} - \lambda & -\frac{3}{5} & 0 \\ \frac{3}{5} & \frac{4}{5} - \lambda & 0 \\ 0 & 0 & 2 - \lambda \end{pmatrix} = \boxed{(2 - \lambda)}\left(\lambda^2 - \frac{8}{5}\lambda + 1 \right).$$

We computed the roots of this polynomial (times 5) before:

$$\lambda = \quad 2, \quad \frac{4 + 3i}{5}, \quad \frac{4 - 3i}{5}.$$

We eyeball an eigenvector with eigenvalue 2 as $(0, 0, 1)$.

$$A = \begin{pmatrix} \frac{4}{5} & -\frac{3}{5} & 0 \\ \frac{3}{5} & \frac{4}{5} & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

To find the other eigenvectors, we row reduce:

$$A - \frac{4+3i}{5}I = \begin{pmatrix} -\frac{3}{5}i & -\frac{3}{5} & 0 \\ \frac{3}{5} & -\frac{3}{5}i & 0 \\ 0 & 0 & 2 - \frac{4+3i}{5} \end{pmatrix} \xrightarrow{\text{scale rows}} \begin{pmatrix} -i & -1 & 0 \\ 1 & -i & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The second row is i times the first:

$$\xrightarrow{\text{row replacement}} \begin{pmatrix} -i & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{\text{divide by } -i,\text{swap}} \begin{pmatrix} 1 & -i & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

The parametric form is $x = iy$, $z = 0$, so an eigenvector is $\begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix}$. Therefore, an

eigenvector with conjugate eigenvalue $\dfrac{4-3i}{5}$ is $\begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix}$.

Summary

- One can do arithmetic with complex numbers just like real numbers: add, subtract, multiply, divide.
- Multiplying complex numbers multiplies the magnitudes and adds the arguments.
- An $n \times n$ matrix always exactly has complex n eigenvalues, counted with (algebraic) multiplicity.
- There's a trick for computing the (complex) eigenspace of a 2×2 matrix:

$$A - \lambda I_2 = \begin{pmatrix} a & b \\ \star & \star \end{pmatrix} \rightsquigarrow v = \begin{pmatrix} -b \\ a \end{pmatrix} \quad \text{(unless } a = b = 0\text{).}$$

- The complex eigenvalues and eigenvectors of a *real* matrix come in complex conjugate pairs:

$$Av = \lambda v \quad \implies \quad A\overline{v} = \overline{\lambda}\overline{v}.$$

Theorem

Let A be a 2×2 matrix with complex (non-real) eigenvalue λ, and let v be an eigenvector. Then

$$A = PCP^{-1}$$

where

$$P = \begin{pmatrix} | & | \\ \operatorname{Re} v & \operatorname{Im} v \\ | & | \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} \operatorname{Re} \lambda & \operatorname{Im} \lambda \\ -\operatorname{Im} \lambda & \operatorname{Re} \lambda \end{pmatrix}.$$

The matrix C is a composition of rotation by $-\arg(\lambda)$ and scaling by $|\lambda|$:

$$C = \begin{pmatrix} |\lambda| & 0 \\ 0 & |\lambda| \end{pmatrix} \begin{pmatrix} \cos(-\arg(\lambda)) & -\sin(-\arg(\lambda)) \\ \sin(-\arg(\lambda)) & \cos(-\arg(\lambda)) \end{pmatrix}.$$

A 2×2 matrix with complex eigenvalue λ is similar to (rotation by the argument of $\overline{\lambda}$) composed with (scaling by $|\lambda|$). This is multiplication by $\overline{\lambda}$ in $\mathbf{C} \sim \mathbf{R}^2$.

What does $A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ do geometrically?

▶ The characteristic polynomial is

$$f(\lambda) = \lambda^2 - \operatorname{Tr}(A)\,\lambda + \det(A) = \lambda^2 - 2\lambda + 2.$$

The roots are

$$\frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i.$$

▶ Let $\lambda = 1 - i$. We compute an eigenvector v:

$$A - \lambda I = \begin{pmatrix} i & -1 \\ \star & \star \end{pmatrix} \rightsquigarrow v = \begin{pmatrix} 1 \\ i \end{pmatrix}.$$

▶ Therefore, $A = PCP^{-1}$ where

$$P = \left(\operatorname{Re}\begin{pmatrix} 1 \\ i \end{pmatrix} \quad \operatorname{Im}\begin{pmatrix} 1 \\ i \end{pmatrix} \right) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$C = \begin{pmatrix} \operatorname{Re}\lambda & \operatorname{Im}\lambda \\ -\operatorname{Im}\lambda & \operatorname{Re}\lambda \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$A = C = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \qquad \lambda = 1 - i$$

▶ The matrix $C = A$ scales by a factor of

$$|\lambda| = \sqrt{1^2 + 1^2} = \sqrt{2}.$$

▶ The argument of λ is $-\pi/4$:
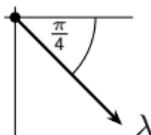
Therefore $C = A$ rotates by $+\pi/4$.

▶ (We already knew this because $A = \sqrt{2}$ times the matrix for rotation by $\pi/4$ from before.)

A
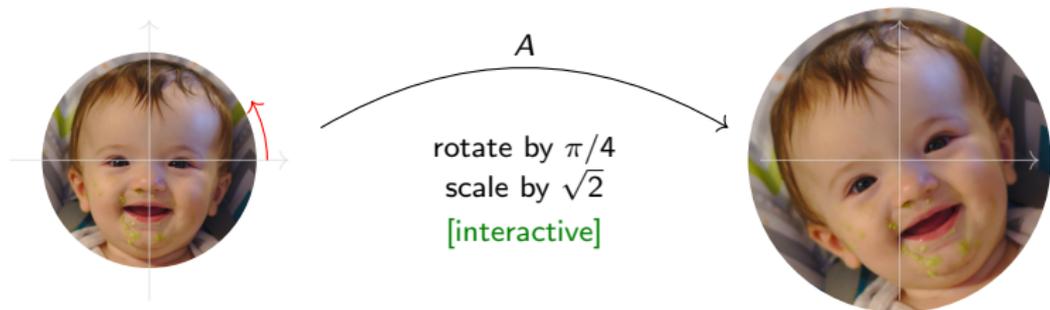
rotate by $\pi/4$
scale by $\sqrt{2}$
[interactive]

What does $A = \begin{pmatrix} \sqrt{3}+1 & -2 \\ 1 & \sqrt{3}-1 \end{pmatrix}$ do geometrically?

▶ The characteristic polynomial is

$$f(\lambda) = \lambda^2 - \mathrm{Tr}(A)\,\lambda + \det(A) = \lambda^2 - 2\sqrt{3}\,\lambda + 4.$$

The roots are

$$\frac{2\sqrt{3} \pm \sqrt{12-16}}{2} = \sqrt{3} \pm i.$$

▶ Let $\lambda = \sqrt{3} - i$. We compute an eigenvector v:

$$A - \lambda I = \begin{pmatrix} 1+i & -2 \\ \star & \star \end{pmatrix} \rightsquigarrow v = \begin{pmatrix} 2 \\ 1+i \end{pmatrix}.$$

▶ It follows that $A = PCP^{-1}$ where

$$P = \begin{pmatrix} \mathrm{Re}\begin{pmatrix} 2 \\ 1+i \end{pmatrix} & \mathrm{Im}\begin{pmatrix} 2 \\ 1+i \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$$

$$C = \begin{pmatrix} \mathrm{Re}\,\lambda & \mathrm{Im}\,\lambda \\ -\mathrm{Im}\,\lambda & \mathrm{Re}\,\lambda \end{pmatrix} = \begin{pmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{pmatrix}.$$

$$A = \begin{pmatrix} \sqrt{3}+1 & -2 \\ 1 & \sqrt{3}-1 \end{pmatrix} \qquad C = \begin{pmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{pmatrix} \qquad \lambda = \sqrt{3} - i$$

▶ The matrix C scales by a factor of

$$|\lambda| = \sqrt{(\sqrt{3})^2 + (-1)^2} = \sqrt{4} = 2.$$

▶ The argument of λ is $-\pi/6$:
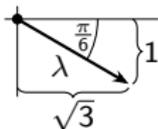
Therefore C rotates by $+\pi/6$.

What does $A = \begin{pmatrix} \sqrt{3}+1 & -2 \\ 1 & \sqrt{3}-1 \end{pmatrix}$ do geometrically?

C

rotate by $\pi/6$
scale by 2
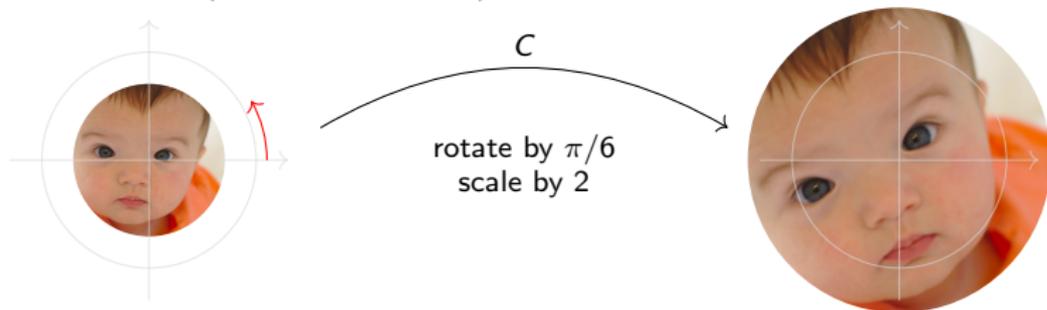
$A = PCP^{-1}$ does the same thing, but with respect to the basis $\mathcal{B} = \left\{ \binom{2}{1}, \binom{0}{1} \right\}$ of columns of P:

A

"rotate around an ellipse"
scale by 2
[interactive]
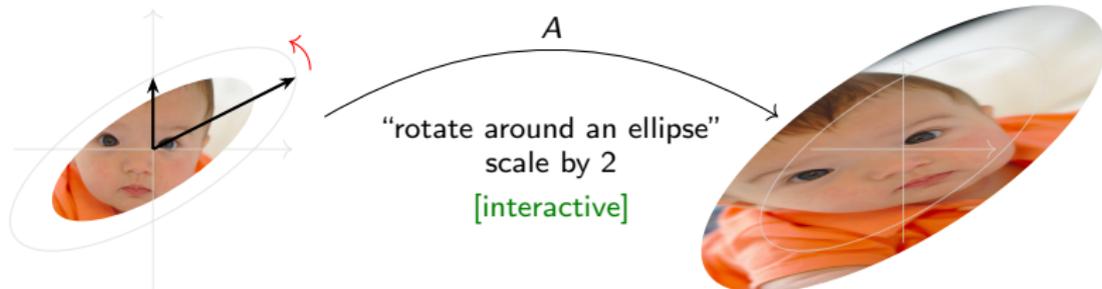
Let A be a real matrix with a complex eigenvalue λ. One way to understand the geometry of A is to consider the difference equation $v_{n+1} = Av_n$, i.e. the sequence of vectors $v, Av, A^2v, \ldots$

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{3}+1 & -2 \\ 1 & \sqrt{3}-1 \end{pmatrix} \qquad A = \frac{1}{2} \begin{pmatrix} \sqrt{3}+1 & -2 \\ 1 & \sqrt{3}-1 \end{pmatrix} \qquad A = \frac{1}{2\sqrt{2}} \begin{pmatrix} \sqrt{3}+1 & -2 \\ 1 & \sqrt{3}-1 \end{pmatrix}$$

$$\lambda = \frac{\sqrt{3}-i}{\sqrt{2}} \qquad\qquad \lambda = \frac{\sqrt{3}-i}{2} \qquad\qquad \lambda = \frac{\sqrt{3}-i}{2\sqrt{2}}$$

$|\lambda| > 1$ $\qquad\qquad\qquad$ $|\lambda| = 1$ $\qquad\qquad\qquad$ $|\lambda| < 1$
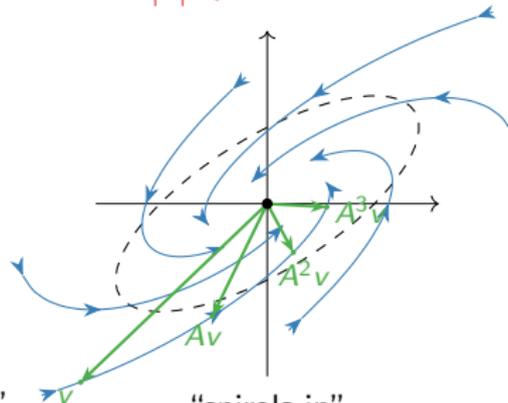
"spirals out" $\qquad\qquad$ "rotates around an ellipse" $\qquad\qquad$ "spirals in"

[interactive] $\qquad\qquad\qquad$ [interactive] $\qquad\qquad\qquad$ [interactive]

Theorem

Let A be a 2×2 matrix with complex (non-real) eigenvalue λ, and let v be an eigenvector. Then

$$A = PCP^{-1}$$

where

$$P = \begin{pmatrix} | & | \\ \text{Re } v & \text{Im } v \\ | & | \end{pmatrix} \quad \text{and} \quad C = (\text{rotation}) \cdot (\text{scaling}).$$

This is very analogous to diagonalization. In the 2×2 case:

Theorem

Let A be a 2×2 matrix with linearly independent eigenvectors v_1, v_2 and associated eigenvalues λ_1, λ_2. Then

$$A = PDP^{-1}$$

where

$$P = \begin{pmatrix} | & | \\ v_1 & v_2 \\ | & | \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

scale x-axis by λ_1
scale y-axis by λ_2

We can draw analogous pictures for a matrix with 2 real eigenvalues.

Example: Let $A = \dfrac{1}{4} \begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix}$.

This has eigenvalues $\lambda_1 = 2$ and $\lambda_2 = \frac{1}{2}$, with eigenvectors

$$v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad v_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Therefore, $A = PDP^{-1}$ with

$$P = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}.$$

So A scales the v_1-direction by 2 and the v_2-direction by $\frac{1}{2}$.

A

scale v_1 by 2
scale v_2 by $\frac{1}{2}$
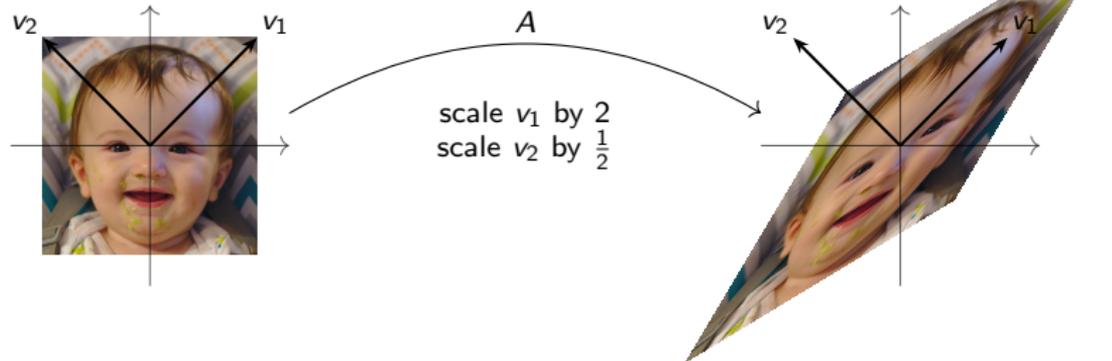
Picture with 2 Real Eigenvalues

We can also draw a picture from the perspective a difference equation: in other words, we draw $v, Av, A^2 v, \ldots$

$$A = \frac{1}{4}\begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix} \qquad \begin{array}{cc} \lambda_1 = 2 & \lambda_2 = \frac{1}{2} \\ |\lambda_1| > 1 & |\lambda_1| < 1 \end{array}$$

[interactive]
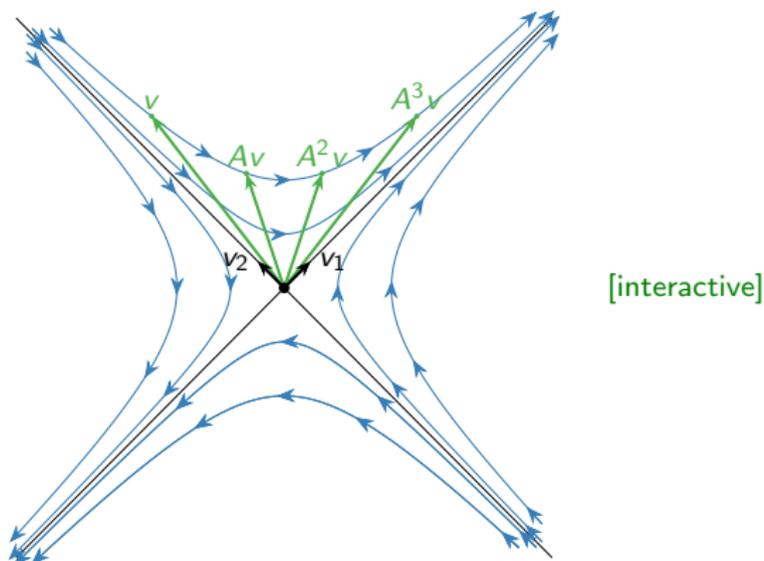
Exercise: Draw analogous pictures when $|\lambda_1|, |\lambda_2|$ are any combination of $< 1, = 1, > 1$.

Theorem

Let A be a real $n \times n$ matrix. Suppose that for each (real or complex) eigenvalue, the dimension of the eigenspace equals the algebraic multiplicity. Then $A = PCP^{-1}$, where P and C are as follows:

1. C is **block diagonal**, where the blocks are 1×1 blocks containing the real eigenvalues (with their multiplicities), or 2×2 blocks containing the matrices $\begin{pmatrix} \operatorname{Re}\lambda & \operatorname{Im}\lambda \\ -\operatorname{Im}\lambda & \operatorname{Re}\lambda \end{pmatrix}$ for each non-real eigenvalue λ (with multiplicity).

2. The columns of P form bases for the eigenspaces for the real eigenvectors, or come in pairs ($\operatorname{Re} v$ $\operatorname{Im} v$) for the non-real eigenvectors.

For instance, if A is a 3×3 matrix with one conjugate pair of complex eigenvalues $\lambda_1, \overline{\lambda}_1$ with eigenvectors $v_1, \overline{v}_1$, and one real eigenvalue λ_2 with eigenvector v_2, then $A = PCP^{-1}$ with

$$P = \begin{pmatrix} | & | & | \\ \operatorname{Re} v_1 & \operatorname{Im} v_1 & v_2 \\ | & | & | \end{pmatrix} \quad C = \begin{pmatrix} \operatorname{Re}\lambda_1 & \operatorname{Im}\lambda_1 & 0 \\ -\operatorname{Im}\lambda_1 & \operatorname{Re}\lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix}$$

Let $C = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$. This acts on the *xy*-plane by rotation by $\pi/4$ and scaling by $\sqrt{2}$. This acts on the *z*-axis by scaling by 2. Pictures:

from above

looking down *y*-axis

[interactive]
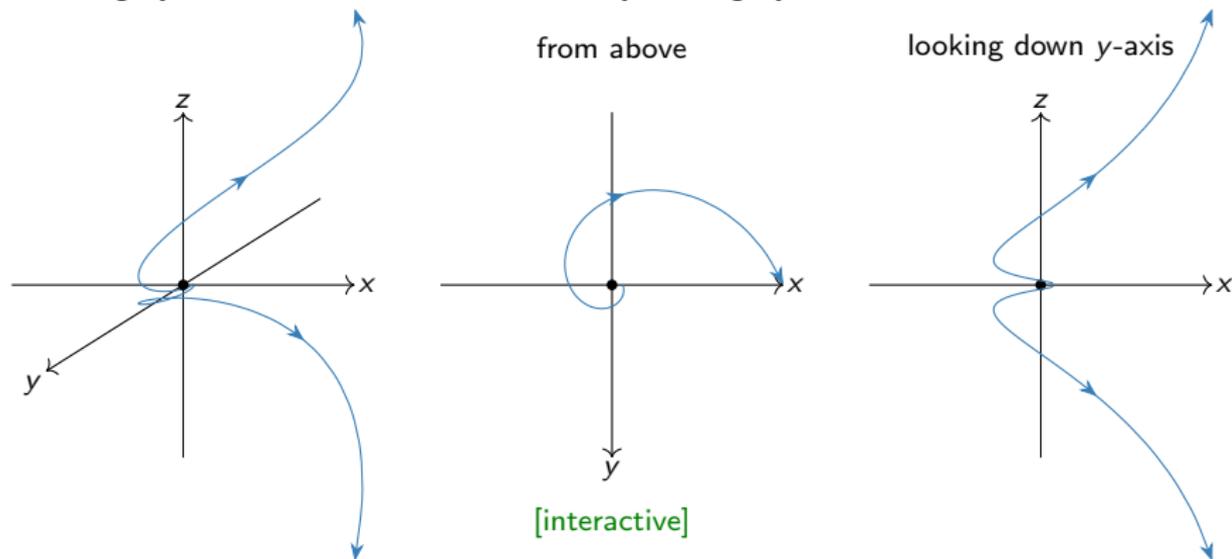
Remember, in general $A = PCP^{-1}$ is only *similar* to such a matrix C: so the x, y, z axes have to be replaced by the columns of P.

Summary

▶ There is a procedure analogous to diagonalization for matrices with complex eigenvalues. In the 2×2 case, the result is

$$A = PCP^{-1}$$

where C is a rotation-scaling matrix.

▶ Multiplication by a 2×2 matrix with a complex eigenvalue λ spirals out if $|\lambda| > 1$, rotates around an ellipse if $|\lambda| = 1$, and spirals in if $|\lambda| < 1$.

▶ There are analogous pictures for 2×2 matrices with real eigenvalues.

▶ For larger matrices, you have to combine diagonalization and "complex diagonalization". You get a block diagonal matrix with scalars and rotation-scaling matrices on the diagonal.