Section 2.9

Dimension and Rank

Coefficients of Basis Vectors

Recall: a **basis** of a subspace V is a set of vectors that *spans* V and is *linearly independent*.

Lemma like a theorem, but less substantial

If $\mathcal{B} = \{v_1, v_2, \ldots, v_m\}$ is a basis for a subspace V, then any vector x in V can be written as a linear combination

$$x = c_1 v_1 + c_2 v_2 + \cdots + c_m v_m$$

for *unique* coefficients $c_1, c_2, \ldots, c_m$.

We know x is a linear combination of the v_i because they span V. Suppose that we can write x as a linear combination with different coefficients:

$$x = c_1 v_1 + c_2 v_2 + \cdots + c_m v_m$$
$$x = c_1' v_1 + c_2' v_2 + \cdots + c_m' v_m$$

Subtracting:

$$0 = x - x = (c_1 - c_1')v_1 + (c_2 - c_2')v_2 + \cdots + (c_m - c_m')v_m$$

Since $v_1, v_2, \ldots, v_m$ are linearly independent, they only have the trivial linear dependence relation. That means each $c_i - c_i' = 0$, or $c_i = c_i'$.

Bases as Coordinate Systems

The unit coordinate vectors $e_1, e_2, \ldots, e_n$ form a basis for $\mathbf{R}^n$. Any vector is a unique linear combination of the e_i:

$$v = \begin{pmatrix} 3 \\ 5 \\ -2 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 5\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - 2\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 3e_1 + 5e_2 - 2e_3.$$

Observe: the *coordinates* of v are exactly the *coefficients* of e_1, e_2, e_3.

We can go backwards: given any basis $\mathcal{B}$, we interpret the coefficients of a linear combination as "coordinates" with respect to $\mathcal{B}$.

Definition

Let $\mathcal{B} = \{v_1, v_2, \ldots, v_m\}$ be a basis of a subspace V. Any vector x in V can be written uniquely as a linear combination $x = c_1v_1 + c_2v_2 + \cdots + c_mv_m$. The coefficients $c_1, c_2, \ldots, c_m$ are the **coordinates of** x **with respect to** $\mathcal{B}$. The $\mathcal{B}$**-coordinate vector of** x is the vector

$$[x]_{\mathcal{B}} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{pmatrix} \quad \text{in } \mathbf{R}^m.$$

In other words, a basis gives a *coordinate system* on V.

Let $v_1 = \begin{pmatrix}1\\0\\1\end{pmatrix}$, $v_2 = \begin{pmatrix}1\\1\\1\end{pmatrix}$, $\mathcal{B} = \{v_1, v_2\}$, $V = \mathrm{Span}\{v_1, v_2\}$.

Verify that $\mathcal{B}$ is a basis:
Span: by definition $V = \mathrm{Span}\{v_1, v_2\}$.
Linearly independent: because they are not multiples of each other.

Question: If $[w]_{\mathcal{B}} = \binom{5}{2}$, then what is w? [interactive]

$$[w]_{\mathcal{B}} = \begin{pmatrix}5\\2\end{pmatrix} \quad \text{means} \quad w = 5v_1 + 2v_2 = 5\begin{pmatrix}1\\0\\1\end{pmatrix} + 2\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}7\\2\\7\end{pmatrix}.$$

Question: Find the $\mathcal{B}$-coordinates of $w = \begin{pmatrix}5\\3\\5\end{pmatrix}$. [interactive]

We have to solve the vector equation $w = c_1v_1 + c_2v_2$ in the unknowns c_1, c_2.

$$\left(\begin{array}{cc|c}1&1&5\\0&1&3\\1&1&5\end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c}1&1&5\\0&1&3\\0&0&0\end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c}1&0&2\\0&1&3\\0&0&0\end{array}\right)$$

So $c_1 = 2$ and $c_2 = 3$, so $w = 2v_1 + 3v_2$ and $[w]_{\mathcal{B}} = \binom{2}{3}$.

Let $v_1 = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}$, $v_2 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$, $v_3 = \begin{pmatrix} 2 \\ 8 \\ 6 \end{pmatrix}$, $\quad V = \mathrm{Span}\{v_1, v_2, v_3\}$.

Question: Find a basis for V. [interactive]

V is the column span of the matrix

$$A = \begin{pmatrix} 2 & -1 & 2 \\ 3 & 1 & 8 \\ 2 & 1 & 6 \end{pmatrix} \xrightarrow{\text{row reduce}} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

A basis for the column span is formed by the pivot columns: $\mathcal{B} = \{v_1, v_2\}$.

Question: Find the $\mathcal{B}$-coordinates of $x = \begin{pmatrix} 4 \\ 11 \\ 8 \end{pmatrix}$. [interactive]

We have to solve $x = c_1 v_1 + c_2 v_2$.

$$\left(\begin{array}{cc|c} 2 & -1 & 4 \\ 3 & 1 & 11 \\ 2 & 1 & 8 \end{array}\right) \xrightarrow{\text{row reduce}} \left(\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right)$$

So $x = 3v_1 + 2v_2$ and $[x]_{\mathcal{B}} = \binom{3}{2}$.

If $\mathcal{B} = \{v_1, v_2, \ldots, v_m\}$ is a basis for a subspace V and x is in V, then

$$\boxed{[x]_{\mathcal{B}} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{pmatrix} \quad \text{means} \quad x = c_1 v_1 + c_2 v_2 + \cdots + c_m v_m.}$$

Finding the $\mathcal{B}$-coordinates for x means solving the vector equation

$$x = c_1 v_1 + c_2 v_2 + \cdots + c_m v_m$$

in the unknowns $c_1, c_2, \ldots, c_m$. This (usually) means row reducing the augmented matrix

$$\left(\begin{array}{cccc|c} | & | & & | & | \\ v_1 & v_2 & \cdots & v_m & x \\ | & | & & | & | \end{array}\right).$$

Question: What happens if you try to find the $\mathcal{B}$-coordinates of x *not* in V?
You end up with an inconsistent system: V is the span of $v_1, v_2, \ldots, v_m$, and if x is not in the span, then $x = c_1 v_1 + c_2 v_2 + \cdots + c_m v_m$ has no solution.

Bases as Coordinate Systems

Picture

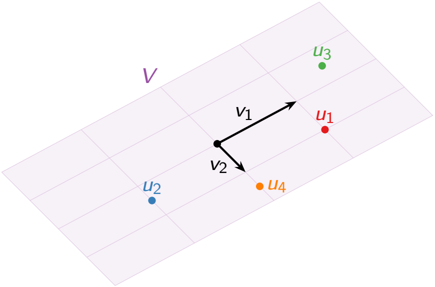
Let

$$v_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \quad v_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

These form a basis $\mathcal{B}$ for the plane

$$V = \mathrm{Span}\{v_1, v_2\}$$

in $\mathbf{R}^3$.



Question: Estimate the $\mathcal{B}$-coordinates of these vectors: [interactive]

$$[u_1]_{\mathcal{B}} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad [u_2]_{\mathcal{B}} = \begin{pmatrix} -1 \\ \frac{1}{2} \end{pmatrix} \qquad [u_3]_{\mathcal{B}} = \begin{pmatrix} \frac{3}{2} \\ -\frac{1}{2} \end{pmatrix} \qquad [u_4]_{\mathcal{B}} = \begin{pmatrix} 0 \\ \frac{3}{2} \end{pmatrix}$$

Choosing a basis $\mathcal{B}$ and using $\mathcal{B}$-coordinates lets us label the points of V with element of $\mathbf{R}^2$.

The Rank Theorem

Recall:

- The **dimension** of a subspace V is the number of vectors in a basis for V.
- A basis for the column space of a matrix A is given by the pivot columns.
- A basis for the null space of A is given by the vectors attached to the free variables in the parametric vector form.

Definition

The **rank** of a matrix A, written $\operatorname{rank} A$, is the dimension of the column space $\operatorname{Col} A$.

Observe:

$$\begin{aligned}\operatorname{rank} A = \dim \operatorname{Col} A &= \text{the number of columns with pivots} \\ \dim \operatorname{Nul} A &= \text{the number of free variables} \\ &= \text{the number of columns without pivots.}\end{aligned}$$

Rank Theorem

If A is an $m \times n$ matrix, then

$$\operatorname{rank} A + \dim \operatorname{Nul} A = n = \text{the number of columns of } A.$$

In other words, [interactive 1] [interactive 2]

(dimension of column span) + (dimension of solution set) = (number of variables).

The Rank Theorem

Example

$$A = \begin{pmatrix} \boxed{\begin{matrix}1\\-2\\2\end{matrix}} & \boxed{\begin{matrix}2\\-3\\4\end{matrix}} & \begin{matrix}0\\4\\0\end{matrix} & \begin{matrix}-1\\5\\-2\end{matrix} \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} \begin{matrix}1\\0\\0\end{matrix} & \begin{matrix}0\\1\\0\end{matrix} & \boxed{\begin{matrix}-8\\4\\0\end{matrix}} & \boxed{\begin{matrix}-7\\3\\0\end{matrix}} \end{pmatrix}$$

basis of Col A free variables

A basis for Col A is

$$\left\{ \begin{pmatrix}1\\-2\\2\end{pmatrix}, \begin{pmatrix}2\\-3\\4\end{pmatrix} \right\},$$

so $\operatorname{rank} A = \dim \operatorname{Col} A = 2$.

Since there are two free variables x_3, x_4, the parametric vector form for the solutions to $Ax = 0$ is

$$x = x_3 \begin{pmatrix}8\\-4\\1\\0\end{pmatrix} + x_4 \begin{pmatrix}7\\-3\\0\\1\end{pmatrix} \xrightarrow{\text{basis for Nul } A} \left\{ \begin{pmatrix}8\\-4\\1\\0\end{pmatrix}, \begin{pmatrix}7\\-3\\0\\1\end{pmatrix} \right\}.$$

Thus $\dim \operatorname{Nul} A = 2$.

The Rank Theorem says $2 + 2 = 4$.

Poll

Poll

Let A and B be 3×3 matrices. Suppose that $\operatorname{rank}(A) = 2$ and $\operatorname{rank}(B) = 2$. Is it possible that $AB = 0$? Why or why not?

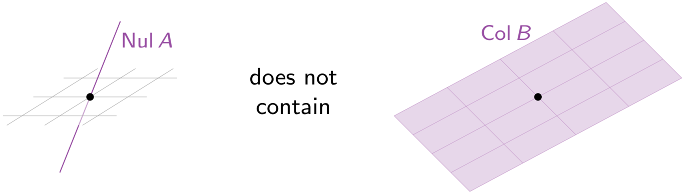
If $AB = 0$, then $ABx = 0$ for every x in $\mathbf{R}^3$.

This means $A(Bx) = 0$, so Bx is in $\operatorname{Nul} A$.

This is true for every x, so $\operatorname{Col} B$ is contained in $\operatorname{Nul} A$.

But $\dim \operatorname{Nul} A = 1$ and $\dim \operatorname{Col} B = 2$, and a 1-dimensional space can't contain a 2-dimensional space.

Hence it can't happen.



The Basis Theorem

Basis Theorem

Let V be a subspace of dimension m. Then:

- Any m linearly independent vectors in V form a basis for V.
- Any m vectors that span V form a basis for V.

Upshot

If you *already* know that $\dim V = m$, and you have m vectors $\mathcal{B} = \{v_1, v_2, \ldots, v_m\}$ in V, then you only have to check *one* of

1. $\mathcal{B}$ is linearly independent, *or*
2. $\mathcal{B}$ spans V

in order for $\mathcal{B}$ to be a basis.

Example: any three linearly independent vectors form a basis for $\mathbf{R}^3$.

The Invertible Matrix Theorem

Let A be an $n \times n$ matrix, and let $T: \mathbf{R}^n \to \mathbf{R}^n$ be the linear transformation $T(x) = Ax$. The following statements are equivalent.

1. A is invertible.
2. T is invertible.
3. A is row equivalent to I_n.
4. A has n pivots.
5. $Ax = 0$ has only the trivial solution.
6. The columns of A are linearly independent.
7. T is one-to-one.
8. $Ax = b$ is consistent for all b in $\mathbf{R}^n$.
9. The columns of A span $\mathbf{R}^n$.
10. T is onto.
11. A has a left inverse (there exists B such that $BA = I_n$).
12. A has a right inverse (there exists B such that $AB = I_n$).
13. A^T is invertible.
14. The columns of A form a basis for $\mathbf{R}^n$.
15. $\operatorname{Col} A = \mathbf{R}^n$.
16. $\dim \operatorname{Col} A = n$.
17. $\operatorname{rank} A = n$.
18. $\operatorname{Nul} A = \{0\}$.
19. $\dim \operatorname{Nul} A = 0$.

These are equivalent to the previous conditions by the Rank Theorem and the Basis Theorem.

Summary

- If $\mathcal{B}$ is a basis for a subspace, we can write a vector in the subspace as a linear combination of the basis vectors, with *unique* coefficients:

$$x = c_1 v_1 + c_2 v_2 + \cdots + c_m v_m.$$

- The coefficients are the $\mathcal{B}$-**coordinates** of x:

$$[x]_{\mathcal{B}} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{pmatrix}.$$

- Finding the $\mathcal{B}$-coordinates means solving the vector equation above.
- The **rank theorem** says the dimension of the column space of a matrix, plus the dimension of the null space, is the number of columns of the matrix.
- The **basis theorem** says that if you already know that $\dim V = m$, and you have m vectors in V, then you only have to check if they span *or* they're linearly independent to know they're a basis.
- There are more conditions of the Invertible Matrix Theorem.