

# Chapter 1

## Linear Equations

Recall that  $\mathbf{R}$  denotes the collection of all real numbers, i.e. the number line. It contains numbers like  $0, -1, \pi, \frac{3}{2}, \dots$

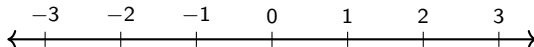
## Definition

Let  $n$  be a positive whole number. We define

$$\mathbf{R}^n = \text{all ordered } n\text{-tuples of real numbers } (x_1, x_2, x_3, \dots, x_n).$$

## Example

When  $n = 1$ , we just get  $\mathbf{R}$  back:  $\mathbf{R}^1 = \mathbf{R}$ . Geometrically, this is the *number line*.

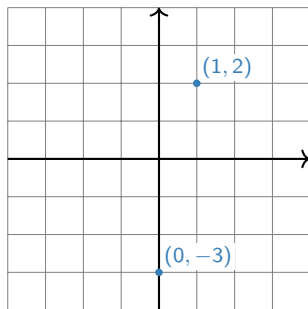


# Line, Plane, Space, ...

Continued

## Example

When  $n = 2$ , we often think of  $\mathbf{R}^2$  as *the  $xy$ -plane* or simply *the plane*. This is because every point on the plane can be represented by an ordered pair of real numbers, namely, its  $x$ - and  $y$ -coordinates.



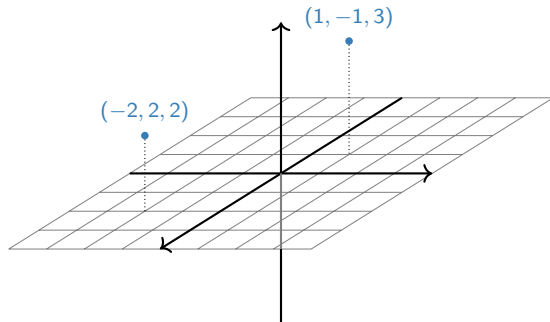
We can use the elements of  $\mathbf{R}^2$  to *label* points on the plane, but  $\mathbf{R}^2$  is not defined to be the  $xy$ -plane!

# Line, Plane, Space, ...

Continued

## Example

When  $n = 3$ , we can think of  $\mathbf{R}^3$  as the *space* we (appear to) live in. This is because every point in space can be represented by an ordered triple of real numbers, namely, its  $x$ -,  $y$ -, and  $z$ -coordinates.

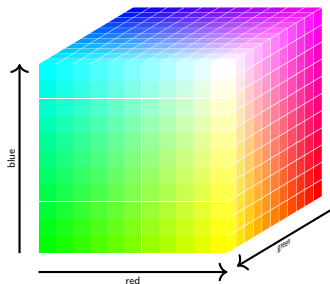


Again, we can use the elements of  $\mathbf{R}^3$  to *label* points in space, but  $\mathbf{R}^3$  is not defined to be space!

## Example

All colors you can see can be described by three quantities: the amount of red, green, and blue light in that color. So we could also think of  $\mathbf{R}^3$  as the space of all *colors*:

$$\mathbf{R}^3 = \text{all colors } (r, g, b).$$



Again, we can use the elements of  $\mathbf{R}^3$  to *label* the colors, but  $\mathbf{R}^3$  is not defined to be the space of all colors!

# Line, Plane, Space, ...

Continued

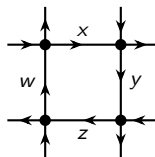
So what is  $\mathbf{R}^4$ ? or  $\mathbf{R}^5$ ? or  $\mathbf{R}^n$ ?

...go back to the *definition*: ordered  $n$ -tuples of real numbers

$$(x_1, x_2, x_3, \dots, x_n).$$

They're still "geometric" spaces, in the sense that our intuition for  $\mathbf{R}^2$  and  $\mathbf{R}^3$  sometimes extends to  $\mathbf{R}^n$ , but they're harder to visualize.

We could use  $\mathbf{R}^4$  to label the amount of traffic  $(x, y, z, w)$  passing through four streets.



We'll make definitions and state theorems that apply to any  $\mathbf{R}^n$ , but we'll only draw pictures for  $\mathbf{R}^2$  and  $\mathbf{R}^3$ .

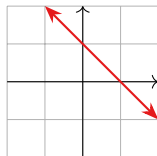
# Section 1.1

## Systems of Linear Equations

# One Linear Equation

What does the solution set of a linear equation look like?

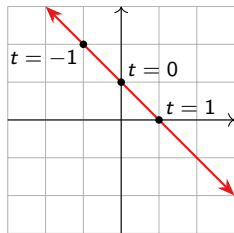
$x + y = 1$   $\rightsquigarrow$  a line in the plane:  $y = 1 - x$   
This is called the **implicit equation** of the line.



We can write the same line in **parametric form** in  $\mathbf{R}^2$ :

$$(x, y) = (t, 1 - t) \quad t \text{ in } \mathbf{R}.$$

This means that every point on the line has the form  $(t, 1 - t)$  for some real number  $t$ .



## Aside

What is a line? A ray that is *straight* and infinite in both directions.

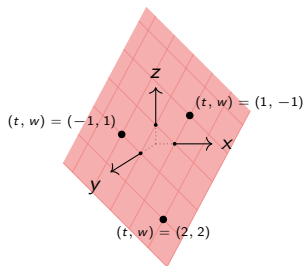


# One Linear Equation

Continued

What does the solution set of a linear equation look like?

$x + y + z = 1$   $\rightsquigarrow$  a plane in space:  
This is the **implicit equation** of the plane.



Does this plane have a **parametric form**?

$$(x, y, z) = (t, w, 1 - t - w) \quad t, w \text{ in } \mathbf{R}.$$

Note you need *two* parameters  $t$  and  $w$ .

## Aside

What is a plane? Intuitively, we think of a plane as a flat sheet of paper that's infinite in all directions. But (as we see on the next slide) this generalizes to something more!

# One Linear Equation

Continued

What does the solution set of a linear equation look like?

$x + y + z + w = 1 \rightsquigarrow$  a “3-plane” in “4-space”... [not pictured here]

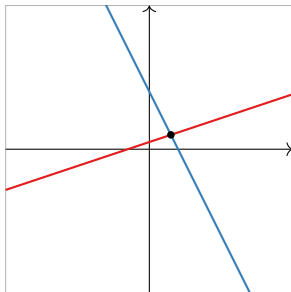
## Systems of Linear Equations

What does the solution set of a *system* of more than one linear equation look like?

$$x - 3y = -3$$

$$2x + y = 8$$

... is the *intersection* of two lines, which is a *point* in this case.



In general it's an intersection of lines, planes, etc.

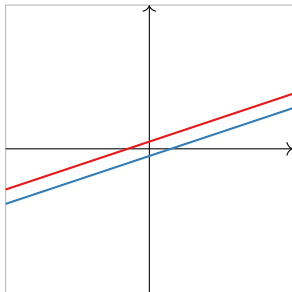
## Kinds of Solution Sets

In what other ways can two lines intersect?

$$x - 3y = -3$$

$$x - 3y = 3$$

has no solution: the lines are  
*parallel*.



A system of equations with no solutions is called **inconsistent**.

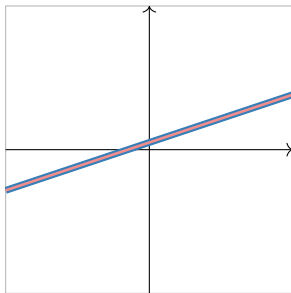
## Kinds of Solution Sets

In what other ways can two lines intersect?

$$x - 3y = -3$$

$$2x - 6y = -6$$

has infinitely many solutions:  
they are the *same line*.



Note that multiplying an equation by a nonzero number gives the *same solution set*. In other words, they are *equivalent* (systems of) equations.

# Solving Systems of Equations

## Example

Solve the system of equations

$$\begin{aligned}x + 2y + 3z &= 6 \\2x - 3y + 2z &= 14 \\3x + y - z &= -2\end{aligned}$$

This is the kind of problem we'll talk about for the first half of the course.

- ▶ A **solution** is a list of numbers  $x, y, z, \dots$  that make *all* of the equations true.
- ▶ The **solution set** is the collection of all solutions.
- ▶ **Solving** the system means finding the solution set.

What is a *systematic* way to solve a system of equations?

# Solving Systems of Equations

## Example

Solve the system of equations

$$x + 2y + 3z = 6$$

$$2x - 3y + 2z = 14$$

$$3x + y - z = -2$$

What strategies do you know?

- ▶ Substitution
- ▶ Elimination

Both are perfectly valid, but only elimination scales well to large numbers of equations.

# Solving Systems of Equations

## Example

Solve the system of equations

$$x + 2y + 3z = 6$$

$$2x - 3y + 2z = 14$$

$$3x + y - z = -2$$

**Elimination method:** in what ways can you manipulate the equations?

- ▶ Multiply an equation by a nonzero number. (scale)
- ▶ Add a multiple of one equation to another. (replacement)
- ▶ Swap two equations. (swap)



# Solving Systems of Equations

## Example

Solve the system of equations

$$x + 2y + 3z = 6$$

$$2x - 3y + 2z = 14$$

$$3x + y - z = -2$$

Multiply first by  $-3$

~~~~~→

$$-3x - 6y - 9z = -18$$

$$2x - 3y + 2z = 14$$

$$3x + y - z = -2$$

Add first to third

~~~~~→

$$-3x - 6y - 9z = -18$$

$$2x - 3y + 2z = 14$$

$$-5y - 10z = -20$$

Now I've eliminated  $x$  from the last equation!

... but there's a long way to go still. Can we make our lives easier?

# Solving Systems of Equations

Better notation

It sure is a pain to have to write  $x, y, z$ , and  $=$  over and over again.

**Matrix notation:** write just the numbers, in a box, instead!

$$\begin{array}{rcl} x + 2y + 3z = & 6 \\ 2x - 3y + 2z = & 14 \\ 3x + y - z = & -2 \end{array} \quad \begin{array}{l} \text{becomes} \\ \rightsquigarrow \end{array} \quad \left( \begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{array} \right)$$

This is called an **(augmented) matrix**. Our equation manipulations become **elementary row operations**:

- ▶ Multiply all entries in a row by a nonzero number. **(scale)**
- ▶ Add a multiple of each entry of one row to the corresponding entry in another. **(row replacement)**
- ▶ Swap two rows. **(swap)**

## Example

Solve the system of equations

$$x + 2y + 3z = 6$$

$$2x - 3y + 2z = 14$$

$$3x + y - z = -2$$

Start:

$$\left( \begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{array} \right)$$

**Goal:** we want our elimination method to eventually produce a system of equations like

$$\begin{array}{rcl} x & = & A \\ y & = & B \\ z & = & C \end{array} \quad \text{or in matrix form,} \quad \left( \begin{array}{ccc|c} 1 & 0 & 0 & A \\ 0 & 1 & 0 & B \\ 0 & 0 & 1 & C \end{array} \right)$$

So we need to do row operations that make the start matrix look like the end one.

**Strategy:** fiddle with it so we only have ones and zeros.

# Row Operations

Continued

$$\left( \begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{array} \right)$$

We want these to be zero.  
So we subtract multiples of the first row.

$$R_2 = R_2 - 2R_1$$

$$R_3 = R_3 - 3R_1$$

$$\left( \begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 3 & 1 & -1 & -2 \end{array} \right)$$

$$\left( \begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 0 & -5 & -10 & -20 \end{array} \right)$$

$$\left( \begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 0 & -5 & -10 & -20 \end{array} \right)$$

We want these to be zero.

It would be nice if this were a 1.  
We could divide by  $-7$ , but that would produce ugly fractions.

Let's swap the last two rows first.

$$R_2 \leftrightarrow R_3$$

$$R_2 = R_2 \div -5$$

$$R_1 = R_1 - 2R_2$$

$$R_3 = R_3 + 7R_2$$

$$\left( \begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -5 & -10 & -20 \\ 0 & -7 & -4 & 2 \end{array} \right)$$

$$\left( \begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & -7 & -4 & 2 \end{array} \right)$$

$$\left( \begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 4 \\ 0 & -7 & -4 & 2 \end{array} \right)$$

$$\left( \begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 10 & 30 \end{array} \right)$$

# Row Operations

Continued

$$\left( \begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 10 & 30 \end{array} \right)$$

We want these to be zero.

Let's make this a 1 first.

$$\begin{array}{l} R_3 = R_3 \div 10 \\ \text{~~~~~} \end{array}$$

$$\begin{array}{l} R_1 = R_1 + R_3 \\ \text{~~~~~} \end{array}$$

$$\begin{array}{l} R_2 = R_2 - 2R_3 \\ \text{~~~~~} \end{array}$$

translates into  
~~~~~

$$\left( \begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 3 \end{array} \right)$$

$$\left( \begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 3 \end{array} \right)$$

$$\left( \begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array} \right)$$

$$\begin{array}{rcl} x & = & 1 \\ y & = & -2 \\ z & = & 3 \end{array}$$

Success!

Check:

$$x + 2y + 3z = 6$$

$$2x - 3y + 2z = 14$$

$$3x + y - z = -2$$

substitute solution  
~~~~~

$$1 + 2 \cdot (-2) + 3 \cdot 3 = 6$$

$$2 \cdot 1 - 3 \cdot (-2) + 2 \cdot 3 = 14$$

$$3 \cdot 1 + (-2) - 3 = -2$$



### Important

The process of doing row operations to a matrix does not change the solution set of the corresponding linear equations!

### Definition

Two matrices are called **row equivalent** if one can be obtained from the other by doing some number of elementary row operations.

So the linear equations of row-equivalent matrices have the *same solution set*.

# A Bad Example

## Example

Solve the system of equations

$$x + y = 2$$

$$3x + 4y = 5$$

$$4x + 5y = 9$$

Let's try doing row operations:

First clear these by subtracting multiples of the first row.

$$\left( \begin{array}{cc|c} 1 & 1 & 2 \\ 3 & 4 & 5 \\ 4 & 5 & 9 \end{array} \right) \begin{array}{l} R_2 = R_2 - 3R_1 \\ R_3 = R_3 - 4R_1 \end{array} \rightarrow \left( \begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 4 & 5 & 9 \end{array} \right)$$
$$\left( \begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{array} \right)$$

Now clear this by subtracting the second row.

$$\left( \begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{array} \right) \begin{array}{l} R_3 = R_3 - R_2 \end{array} \rightarrow \left( \begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{array} \right)$$

# A Bad Example

Continued

$$\left( \begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{array} \right) \begin{array}{l} \text{translates into} \\ \text{~~~~~\rightsquigarrow} \end{array} \begin{array}{l} x + y = 2 \\ y = -1 \\ 0 = 2 \end{array}$$

In other words, the original equations

$$\begin{array}{l} x + y = 2 \\ 3x + 4y = 5 \\ 4x + 5y = 9 \end{array} \quad \text{have the same solutions as} \quad \begin{array}{l} x + y = 2 \\ y = -1 \\ 0 = 2 \end{array}$$

But the latter system obviously has no solutions (there is no way to make them all true), so our original system has no solutions either.

## Definition

A system of equations is called **inconsistent** if it has no solution. It is **consistent** otherwise.