Math 1553 Worksheet §6.4, 6.5

1. Answer yes, no, or maybe. Justify your answers. In each case, A is a matrix whose entries are real numbers.

a) If A is a 3×3 matrix with characteristic polynomial $-\lambda(\lambda - 5)^2$, then the 5-eigenspace is 2-dimensional.

b) If A is an invertible 2×2 matrix, then A is diagonalizable.

c) A 3×3 matrix A can have a non-real complex eigenvalue with multiplicity 2.

d) Suppose A is a 7×7 matrix with four distinct eigenvalues. If one eigenspace has dimension 2, while another eigenspace has dimension 3, then A must be diagonalizable.

Solution.

a) Maybe. The geometric multiplicity of $\lambda = 5$ can be 1 or 2. For example, the matrix $\begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ has a 5- eigenspace which is 2-dimensional, whereas the matrix $\begin{pmatrix} 5 & 1 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ has a 5-eigenspace which is 1-dimensional. Both matrices have characteristic polynomial $-\lambda(5-\lambda)^2$.

b) Maybe. The identity matrix is invertible and diagonalizable, but the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is invertible but not diagonalizable.

c) No. If c is a (non-real) complex eigenvalue with multiplicity 2, then its conjugate $\bar{c}$ is an eigenvalue with multiplicity 2 since complex eigenvalues always occur in conjugate pairs. This would mean A has a characteristic polynomial of degree 4 or more, which is impossible for a 3×3 matrix.

d) Yes. It is a general fact that every eigenvalue of a matrix has a corresponding eigenspace which is at least 1-dimensional. Given this and the fact that A has four total eigenvalues, we see the sum of dimensions of the eigenspaces of A is at least $2+3+1+1 = 7$, and in fact must equal 7 since that is the max possible for a 7×7 matrix. Therefore, A has 7 linearly independent eigenvectors and is therefore diagonalizable.

2. $A = \begin{pmatrix} 2 & 3 & 1 \\ 3 & 2 & 4 \\ 0 & 0 & -1 \end{pmatrix}$.

a) Find the eigenvalues of A, and find a basis for each eigenspace.

b) Is A diagonalizable? If your answer is yes, find a diagonal matrix D and an invertible matrix C so that $A = CDC^{-1}$. If your answer is no, justify why A is not diagonalizable.

Solution.

a) We solve $0 = \det(A - \lambda I)$.

$$0 = \det\begin{pmatrix} 2-\lambda & 3 & 1 \\ 3 & 2-\lambda & 4 \\ 0 & 0 & -1-\lambda \end{pmatrix} = (-1-\lambda)(-1)^6 \det\begin{pmatrix} 2-\lambda & 3 \\ 3 & 2-\lambda \end{pmatrix} = (-1-\lambda)((2-\lambda)^2 - 9)$$

$$= (-1-\lambda)(\lambda^2 - 4\lambda - 5) = -(\lambda+1)^2(\lambda-5).$$

So $\lambda = -1$ and $\lambda = 5$ are the eigenvalues.

$\underline{\lambda = -1}$: $\left(A + I \mid 0\right) = \left(\begin{array}{ccc|c} 3 & 3 & 1 & 0 \\ 3 & 3 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_2 = R_2 - R_1} \left(\begin{array}{ccc|c} 3 & 3 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow[\text{then } R_1 = R_1/3]{R_1 = R_1 - R_2}$

$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$, with solution $x_1 = -x_2$, $x_2 = x_2$, $x_3 = 0$. The (-1)-eigenspace has basis $\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right\}$.

$\underline{\lambda = 5}$:

$$\left(A - 5I \mid 0\right) = \left(\begin{array}{ccc|c} -3 & 3 & 1 & 0 \\ 3 & -3 & 4 & 0 \\ 0 & 0 & -6 & 0 \end{array}\right) \xrightarrow[R_3 = R_3/(-6)]{R_2 = R_2 + R_1} \left(\begin{array}{ccc|c} -3 & 3 & 1 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \xrightarrow[\text{then } R_2 \leftrightarrow R_3,\ R_1/(-3)]{R_1 = R_1 - R_3,\ R_2 = R_2 - 5R_3} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right),$$

with solution $x_1 = x_2,\ x_2 = x_2,\ x_3 = 0$. The 5-eigenspace has basis $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$.

b) A is a 3×3 matrix that only admits 2 linearly independent eigenvectors, so A is not diagonalizable.