MATH 1553, JANKOWSKI (A1-A6)
MIDTERM 3, FALL 2018

Name		**GT Email**	@gatech.edu

Write your section number here: ____________

Please **read all instructions** carefully before beginning.

- Please leave your GT ID card on your desk until your TA matches your exam.
- The maximum score on this exam is 50 points, and you have 50 minutes to complete this exam.
- There are no calculators or aids of any kind (notes, text, etc.) allowed.
- As always, RREF means "reduced row echelon form."
- Show your work, unless instructed otherwise. A correct answer without appropriate work will receive little or no credit! If you cannot fit your work on the front side of the page, use the back side of the page and indicate that you are using the back side.
- We will hand out loose scrap paper, but it **will not be graded** under any circumstances. All work must be written on the exam itself.
- You may cite any theorem proved in class or in the sections we covered in the text.
- Good luck!

This page was intentionally left blank.

Problem 1.

[2 points each]

Answer true if the statement is *always* true. Otherwise, answer false. You do not need to justify your answer. In every case, A is a matrix whose entries are real numbers.

a) **T** **F** If A is a 3×3 matrix and $Ae_1 = Ae_3$, then $\det(A) = 0$.

b) **T** **F** Suppose an $n \times n$ matrix A has n linearly independent columns. Then 0 is not an eigenvalue of A.

c) **T** **F** Suppose a 3×3 matrix A has characteristic polynomial $-\lambda^3 + \lambda$. Then A must be diagonalizable.

d) **T** **F** If A is a 2×2 matrix and $\det(A) = \det(-A)$, then A is not invertible.

e) **T** **F** If A is an $n \times n$ matrix and $\text{Nul}(A-3I) \neq \{0\}$, then $\lambda = 3$ must be an eigenvalue of A.

Solution.

a) True. Two columns of A are equal, so $\det(A) = 0$. Alternatively, $T(x) = Ax$ is not one-to-one since $T(e_1) = T(e_3)$ so A is not invertible hence $\det(A) = 0$.

b) True. The columns of A span $\mathbf{R}^n$ so A is invertible, hence 0 is not an eigenvalue of A.

c) True. $-\lambda^3 + \lambda = -\lambda(\lambda^2 - 1) = -\lambda(\lambda - 1)(\lambda + 1)$ so the 3×3 A has three distinct real eigenvalues and is therefore diagonalizable.

d) False: $\det(-A) = (-1)^2 \det(A) = \det(A)$ for all A, so for example $\det(I) = \det(-I)$.

e) True: $(A-3I)v = 0$ has a solution aside of the zero vector, and therefore $Av = 3v$ for some non-zero vector v.

Scrap paper for problem 1

Problem 2.

[9 points]

Short answer. In this problem, you don't need to justify your answers except in (d), and all entries of any matrix A are real numbers.

a) Complete the following definition (be precise!): Suppose A is an $n \times n$ matrix. We say that a real number λ is an *eigenvalue* of A if...

b) Suppose A is a 3×3 matrix with real entries, and suppose that $\lambda = 1$ and $\lambda = 3$ are eigenvalues of A. For each statement below, circle: YES if it must be true; NO if it must be false; MAYBE if it is sometimes true and sometimes false.

(i) A is diagonalizable. YES NO MAYBE

(ii) $2 - 3i$ is an eigenvalue of A. YES NO MAYBE

(iii) A is invertible. YES NO MAYBE

c) Write a 3×3 matrix A with $\lambda = 5$ as an eigenvalue, so that the 5-eigenspace is the z-axis.

d) Let A be the matrix for reflection across the line $y = 3x$ in $\mathbf{R}^2$.
Is A diagonalizable? Justify your answer.

Solution.

a) ... $Av = \lambda v$ has a non-trivial solution.
(equivalently: $Av = \lambda v$ for some non-zero v in $\mathbf{R}^n$, etc.)

b) (i) Maybe : for example, $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ is diagonalizable while $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix}$ is not.

(ii) No: If $2 - 3i$ is an eigenvalue then so is $2 + 3i$, and that would mean our 3×3 matrix has 4 distinct eigenvalues.

(iii) Maybe. A might have 0 as a third eigenvalue, but it might not. For example, $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ vs. $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.

c) Many examples are possible. For example, $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{pmatrix}$.

d) Yes because A is a 2×2 matrix with two distinct eigenvalues: A has eigenvalues 1 and -1 just like reflection across *any* line through the origin in $\mathbf{R}^2$: it fixes vectors on the line $y = 3x$ and flips vectors perpendicular to $y = 3x$.

Space for extra work on problem 2

Problem 3. [11 points]

Parts (a) and (b) are unrelated.

a) Let $A = \begin{pmatrix} -2 & 2 \\ -5 & 4 \end{pmatrix}$. Find the eigenvalues of A. For the eigenvalue with positive imaginary part, find a corresponding eigenvector. Simplify your answers.

b) Let $B = \begin{pmatrix} 1 & 4 \\ 2 & 5 \end{pmatrix}$ and define a matrix transformation by $T(x) = Bx$. Find the area of $T(S)$, where S is the triangle with vertices $(-1,1)$, $(2,3)$, and $(5,2)$.

Solution.

a) The characteristic polynomial is

$$\det(A - \lambda I) = \lambda^2 - \mathrm{Tr}(A)I + \det(A) = \lambda^2 - 2\lambda + 2$$

which has roots $\lambda = \dfrac{2 \pm \sqrt{4-8}}{2} = \dfrac{2 \pm 2i}{2} = 1 \pm i$, so $\boxed{\lambda_1 = 1 + i, \quad \lambda_2 = 1 - i}$.

For $\lambda_1 = 1 + i$ we get

$$\left(A - \lambda_1 I \mid 0\right) = \left(\begin{array}{cc|c} -3-i & 2 & 0 \\ \star & \star & 0 \end{array}\right) \xrightarrow{R_1 = R_1/(-3-i)} \left(\begin{array}{cc|c} 1 & \frac{2}{-3-i} & 0 \\ \star & \star & 0 \end{array}\right) = \left(\begin{array}{cc|c} 1 & -3/5 + i/5 & 0 \\ 0 & 0 & 0 \end{array}\right).$$

So $x_1 = (\frac{3}{5} - \frac{i}{5})x_2$. An eigenvector is $\begin{pmatrix} \frac{3}{5} - \frac{i}{5} \\ 1 \end{pmatrix}$.

Alternative: The first line of $A - \lambda_1 I$ is $(a \ \ b) = \begin{pmatrix} -3-i & 2 \end{pmatrix}$ so an eigenvector is $\begin{pmatrix} -b \\ a \end{pmatrix} = \begin{pmatrix} -2 \\ -3-i \end{pmatrix}$.

b) Using base point $(-1,1)$, the area of the triangle is

$$\frac{1}{2}\left|\det\begin{pmatrix} 3 & 6 \\ 2 & 1 \end{pmatrix}\right| = \frac{1}{2}(9) = \frac{9}{2}.$$

$$\mathrm{Area}(T(S)) = |\det(B)|\mathrm{Area}(S) = |5-8|\frac{9}{2} = \frac{27}{2}.$$

Space for extra work on problem 3

Problem 4.

[10 points]

Consider the matrix $A = \begin{pmatrix} 2 & 1 & -1 \\ 0 & 0 & 2 \\ 0 & -1 & 3 \end{pmatrix}$.

a) Find the characteristic polynomial of A and the eigenvalues of A.

b) For each eigenvalue of A, find a basis for the corresponding eigenspace.

c) Is A diagonalizable? If yes, find an invertible matrix C and a diagonal matrix D so that $A = CDC^{-1}$. If no, justify why A is not diagonalizable.

Solution.

a) Cofactor expansion along the first column gives

$$\det(A - \lambda I) = \det\begin{pmatrix} 2-\lambda & 1 & -1 \\ 0 & -\lambda & 2 \\ 0 & -1 & 3-\lambda \end{pmatrix} = (2-\lambda)\Big[(-\lambda)(3-\lambda)+2\Big]$$

$$= (2-\lambda)\Big[\lambda^2 - 3\lambda + 2\Big] = -(\lambda-2)\Big[(\lambda-2)(\lambda-1)\Big] = -(\lambda-2)^2(\lambda-1).$$

The eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 2$.

b)

$$\big(A - I \mid 0\big) = \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 2 & 0 \end{array}\right) \xrightarrow[R_1 = R_1 + R_2,\text{ then } R_2 = -R_2]{R_3 = R_3 - R_2} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

So $x_1 = -x_3$, $x_2 = 2x_3$, and x_3 is free. A basis for the 1-eigenspace is $\left\{\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}\right\}$.

$$\big(A - 2I \mid 0\big) = \left(\begin{array}{ccc|c} 0 & 1 & -1 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right) \xrightarrow{RREF} \left(\begin{array}{ccc|c} 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

So x_1 and x_3 are free and $x_2 = x_3$. A basis for the 2-eigenspace is $\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.\right\}$

c) We've found 3 linearly independent eigenvectors, so A is diagonalizable: $A = CDC^{-1}$ where

$$C = \begin{pmatrix} -1 & 1 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

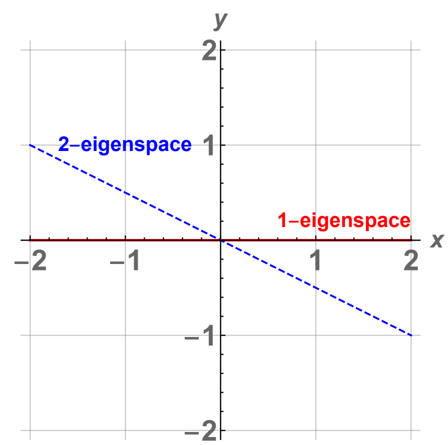
Space for extra work on problem 4

Problem 5.

[10 points]

Parts (a) and (b) are unrelated.

a) Let $A = \begin{pmatrix} 1 & -2 \\ 0 & 2 \end{pmatrix}$. Draw the eigenspaces of A below. Clearly label each eigenspace.



b) Find the matrix A whose (-2)-eigenspace is the y-axis in $\mathbf{R}^2$ and whose 3-eigenspace is the line $y = 4x$.

Solution.

a) The matrix is upper-triangular so the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 2$.

$$\left(A - I \mid 0\right) = \left(\begin{array}{cc|c} 0 & -2 & 0 \\ 0 & 1 & 0 \end{array}\right) \xrightarrow{RREF} \left(\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

So the 1-eigenspace has x free and $y = 0$. This is the x-axis.

$$\left(A - 2I \mid 0\right) = \left(\begin{array}{cc|c} -1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

so the 2-eigenspace is the line $x = -2y$, thus $y = \frac{-x}{2}$.

b) $A = CDC^{-1}$ where $C = \begin{pmatrix} 0 & 1 \\ 1 & 4 \end{pmatrix}$ and $D = \begin{pmatrix} -2 & 0 \\ 0 & 3 \end{pmatrix}$. We find $C^{-1} = \frac{1}{-1}\begin{pmatrix} 4 & -1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -4 & 1 \\ 1 & 0 \end{pmatrix}$ so

$$A = CDC^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} -2 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} -4 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} 8 & -2 \\ 3 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 20 & -2 \end{pmatrix}.$$

Space for extra work on problem 5