Math 1553 Supplement §6.4, 6.5

For those who want additional practice problems after completing the worksheet, here are some extra practice problems.

1. **a)** If A is the matrix that implements rotation by 143° in $\mathbf{R}^2$, then A has no real eigenvalues.

b) If A is diagonalizable and invertible, then A^{-1} is diagonalizable.

c) A 3×3 (real) matrix can have eigenvalues $3, 5$, and $2+i$.

Solution.

a) True. If A had a real eigenvalue λ, then we would have $Ax = \lambda x$ for some vector x in $\mathbf{R}^2$. This means that x would lie on the same line through the origin as the rotation of x by 143°, which is impossible.

b) True. If $A = CDC^{-1}$ and A is invertible then its eigenvalues are all nonzero, so the diagonal entries of D are nonzero and thus D is invertible (pivot in every diagonal position). Thus, $A^{-1} = (CDC^{-1})^{-1} = (C^{-1})^{-1}D^{-1}C^{-1} = CD^{-1}C^{-1}$.

c) False. If $2+i$ is an eigenvalue then so is its conjugate $2-i$.

2. Let $A = \begin{pmatrix} 8 & 36 & 62 \\ -6 & -34 & -62 \\ 3 & 18 & 33 \end{pmatrix}$.

The characteristic polynomial for A is $-\lambda^3 + 7\lambda^2 - 16\lambda + 12$, and $\lambda - 3$ is a factor. Decide if A is diagonalizable. If it is, find an invertible matrix C and a diagonal matrix D such that $A = CDC^{-1}$.

Solution.

By polynomial division,

$$\frac{-\lambda^3 + 7\lambda^2 - 16\lambda + 12}{\lambda - 3} = -\lambda^2 + 4\lambda - 4 = -(\lambda-2)^2.$$

Thus, the characteristic poly factors as $-(\lambda-3)(\lambda-2)^2$, so the eigenalues are $\lambda_1 = 3$ and $\lambda_2 = 2$.

For $\lambda_1 = 3$, we row-reduce $A - 3I$:

$$\begin{pmatrix} 5 & 36 & 62 \\ -6 & -37 & -62 \\ 3 & 18 & 30 \end{pmatrix} \xrightarrow[\text{(New } R_1)/3]{R_1 \leftrightarrow R_3} \begin{pmatrix} 1 & 6 & 10 \\ -6 & -37 & -62 \\ 5 & 36 & 62 \end{pmatrix} \xrightarrow[R_3 = R_3 - 5R_1]{R_2 = R_2 + 6R_1} \begin{pmatrix} 1 & 6 & 10 \\ 0 & -1 & -2 \\ 0 & 6 & 12 \end{pmatrix}$$

$$\xrightarrow[\text{then } R_2 = -R_2]{R_3 = R_3 + 6R_2} \begin{pmatrix} 1 & 6 & 10 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1 = R_1 - 6R_2} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

Therefore, the solutions to $\big(A-3I \mid 0\big)$ are $x_1 = 2x_3,\ \ x_2 = -2x_3,\ \ x_3 = x_3$.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2x_3 \\ -2x_3 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}. \qquad \text{The 3-eigenspace has basis } \left\{ \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \right\}.$$

For $\lambda_2 = 2$, we row-reduce $A-2I$:

$$\begin{pmatrix} 6 & 36 & 62 \\ -6 & -36 & -62 \\ 3 & 18 & 31 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 6 & \frac{31}{3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The solutions to $\big(A-2I \quad 0\big)$ are $x_1 = -6x_2 - \frac{31}{3}x_3,\ x_2 = x_2,\ x_3 = x_3$.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -6x_2 - \frac{31}{3}x_3 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \begin{pmatrix} -6 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -\frac{31}{3} \\ 0 \\ 1 \end{pmatrix}.$$

The 2-eigenspace has basis $\left\{ \begin{pmatrix} -6 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{31}{3} \\ 0 \\ 1 \end{pmatrix} \right\}$.

Therefore, $A = CDC^{-1}$ where

$$C = \begin{pmatrix} 2 & -6 & -\frac{31}{3} \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Note that we arranged the eigenvectors in C in order of the eigenvalues $3,\ 2,\ 2$, so we had to put the diagonals of D in the same order.

3. Give examples of 2×2 matrices with the following properties. Justify your answers.

a) A matrix A which is invertible and diagonalizable.

b) A matrix B which is invertible but not diagonalizable.

c) A matrix C which is not invertible but is diagonalizable.

d) A matrix D which is neither invertible nor diagonalizable.

Solution.

a) We can take any diagonal matrix with nonzero diagonal entries:

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

b) A shear has only one eigenvalue $\lambda = 1$. The associated eigenspace is the x-axis, so there do not exist two linearly independent eigenvectors. Hence it is not diagonalizable.

$$B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

c) We can take any diagonal matrix with some zero diagonal entries:

$$C = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

d) Such a matrix can only have the eigenvalue zero — otherwise it would have two eigenvalues, hence be diagonalizable. Thus the characteristic polynomial is $f(\lambda) = \lambda^2$. Here is a matrix with trace and determinant zero, whose zero-eigenspace (i.e., null space) is not all of $\mathbf{R}^2$:

$$D = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

4. Let $A = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$. Find all eigenvalues of A. For each eigenvalue, find an associated eigenvector.

Solution.

The characteristic polynomial is

$$\lambda^2 - \text{Tr}(A)\lambda + \det(A) = \lambda^2 - 2\lambda + 5$$

$$\lambda^2 - 2\lambda + 5 = 0 \iff \lambda = \frac{2 \pm \sqrt{4-20}}{2} = \frac{2 \pm 4i}{2} = 1 \pm 2i.$$

For $\lambda_1 = 1 - 2i$, we find an eigenvector. For $A - (1-2i)I$, the second row must be a multiple of the first since $A - (1-2i)I$ is a non-invertible 2×2 matrix, so row-reduction will automatically destroy the second row.

$$\big(A - (1-2i)I \mid 0\big) = \left(\begin{array}{cc|c} 2i & 2 & 0 \\ -2 & 2i & 0 \end{array}\right) \xrightarrow[\text{then } R_1 = R_1/(2i)]{R_2 = R_2 - iR_1} \left(\begin{array}{cc|c} 1 & -i & 0 \\ 0 & 0 & 0 \end{array}\right).$$

So $x_1 = ix_2$ and x_2 is free. An eigenvector is $v_1 = \begin{pmatrix} i \\ 1 \end{pmatrix}$.

An eigenvector for $\lambda_2 = 1 + i$ is $v_2 = \overline{v_1} = \begin{pmatrix} -i \\ 1 \end{pmatrix}$.

Alternatively: for the eigenvalue $\lambda = 1 - 2i$, we can use a trick you may have seen in class: the first row $\begin{pmatrix} a & b \end{pmatrix}$ of $A - \lambda I$ will lead to an eigenvector $\begin{pmatrix} -b \\ a \end{pmatrix}$ (or equivalently, $\begin{pmatrix} b \\ -a \end{pmatrix}$ if you prefer).

$$\big(A - (1-2i)I \mid 0\big) = \left(\begin{array}{cc|c} 2i & 2 & 0 \\ (*) & (*) & 0 \end{array}\right) \implies v = \begin{pmatrix} -2 \\ 2i \end{pmatrix}.$$

Note that this choice of v looks much different than the vector v_1 above, but they are actually equivalent since they are (complex) scalar multiples of each other, as $v = 2iv_1$. From the correspondence between conjugate eigenvalues and their eigenvectors, we know (without doing any additional work!) that for the eigenvalue $\lambda = 1+2i$, a corresponding eigenvector is $w = \overline{v} = \begin{pmatrix} -2 \\ -2i \end{pmatrix}$.

5. Suppose a 2×2 matrix A has eigenvalue $\lambda_1 = -2$ with eigenvector $v_1 = \begin{pmatrix} 3/2 \\ 1 \end{pmatrix}$, and eigenvalue $\lambda_2 = -1$ with eigenvector $v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

a) Find A.

b) Find A^{100}.

Solution.

a) We have $A = CDC^{-1}$ where

$$C = \begin{pmatrix} 3/2 & 1 \\ 1 & -1 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}.$$

We compute $C^{-1} = \dfrac{1}{-5/2}\begin{pmatrix} -1 & -1 \\ -1 & 3/2 \end{pmatrix} = \dfrac{1}{5}\begin{pmatrix} 2 & 2 \\ 2 & -3 \end{pmatrix}$.

$$A = CDC^{-1} = \frac{1}{5}\begin{pmatrix} 3/2 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 2 & 2 \\ 2 & -3 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} -8 & -3 \\ -2 & -7 \end{pmatrix}.$$

b)

$$\begin{aligned} A^{100} = CD^{100}C^{-1} &= \frac{1}{5}\begin{pmatrix} 3/2 & 1 \\ 1 & -1 \end{pmatrix} \cdot D^{100}\begin{pmatrix} 2 & 2 \\ 2 & -3 \end{pmatrix} \\ &= \frac{1}{5}\begin{pmatrix} 3/2 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 2^{100} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 2 \\ 2 & -3 \end{pmatrix} \\ &= \frac{1}{5}\begin{pmatrix} 3/2 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 2 \cdot 2^{100} & 2 \cdot 2^{100} \\ 2 & -3 \end{pmatrix} \\ &= \frac{1}{5}\begin{pmatrix} 3 \cdot 2^{100} + 2 & 3 \cdot 2^{100} - 3 \\ 2^{101} - 2 & 2^{101} + 3 \end{pmatrix}. \end{aligned}$$