Math 1553 Supplement §4.5, 5.1-5.3

Solutions

1. **a)** Fill in: A and B are invertible $n \times n$ matrices, then the inverse of AB is ________.

b) If the columns of an $n \times n$ matrix Z are linearly independent, is Z necessarily invertible? Justify your answer.

c) If A and B are $n \times n$ matrices and $ABx = 0$ has a unique solution, does $Ax = 0$ necessarily have a unique solution? Justify your answer.

Solution.

a) $(AB)^{-1} = B^{-1}A^{-1}$.

b) Yes. The transformation $x \to Zx$ is one-to-one since the columns of Z are linearly independent. Thus Z has a pivot in all n columns, so Z has n pivots. Since Z also has n rows, this means that Z has a pivot in every row, so $x \to Zx$ is onto. Therefore, Z is invertible.

Alternatively, since Z is an $n \times n$ matrix whose columns are linearly independent, the Invertible Matrix Theorem (2.3) in 2.3 says that Z is invertible.

c) Yes. Since AB is an $n \times n$ matrix and $ABx = 0$ has a unique solution, the Invertible Matrix Theorem says that AB is invertible. Note A is invertible and its inverse is $B(AB)^{-1}$, since these are square matrices and

$$A(B(AB)^{-1}) = AB(AB)^{-1} = I_n.$$

Since A is invertible, $Ax = 0$ has a unique solution by the Invertible Matrix Theorem.

2. Let A be an $n \times n$ matrix.

a) Using cofactor expansion, explain why $\det(A) = 0$ if A has a row or a column of zeros.

b) Using cofactor expansion, explain why $\det(A) = 0$ if A has adjacent identical columns.

Solution.

a) If A has zeros for all entries in row i (so $a_{i1} = a_{i2} = \cdots = a_{in} = 0$), then the cofactor expansion along row i is

$$\det(A) \ = \ a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in} \ = \ 0 \cdot C_{i1} + 0 \cdot C_{i2} + \cdots + 0 \cdot C_{in} = 0.$$

Similarly, if A has zeros for all entries in column j, then the cofactor expansion along column j is the sum of a bunch of zeros and is thus 0.

b) If A has identical adjacent columns, then the cofactor expansions will be identical except that one expansion's terms for $\det(A)$ will have plus signs where the other expansion's terms for $\det(A)$ have minus signs (due to the $(-1)^{\text{power}}$ factors) and vice versa.

Therefore, $\det(A) = -\det(A)$, so $\det A = 0$.

3. Find the volume of the parallelepiped in $\mathbf{R}^4$ naturally determined by the vectors

$$\begin{pmatrix}4\\1\\3\\8\end{pmatrix}, \begin{pmatrix}0\\7\\0\\3\end{pmatrix}, \begin{pmatrix}0\\2\\0\\1\end{pmatrix}, \begin{pmatrix}5\\-5\\0\\7\end{pmatrix}.$$

Solution.

We put the vectors as columns of a matrix A and find $|\det(A)|$. For this, we expand $\det(A)$ along the third row because it only has one nonzero entry.

$$\det(A) = 3(-1)^{3+1} \cdot \det\begin{pmatrix}0 & 0 & 5\\7 & 2 & -5\\3 & 1 & 7\end{pmatrix} = 3 \cdot 5(-1)^{1+3}\det\begin{pmatrix}7 & 2\\3 & 1\end{pmatrix} = 3(5)(1)(7-6) = 15.$$

(In the second step, we used the cofactor expansion along the first row since it had only one nonzero entry.)

The volume is $|\det(A)| = |15| = 15$.

4. If A is a 3×3 matrix and $\det(A) = 1$, what is $\det(-2A)$?

Solution.

By determinant properties, scaling one row by c multiplies the determinant by c. When we take cA for an $n\times n$ matrix A, we are multiplying *each* row by c. This multiplies the determinant by c a total of n times.

Thus, if A is $n\times n$, then $\det(cA) = c^n \det(A)$. Here $n = 3$, so

$$\det(-2A) = (-2)^3 \det(A) = -8\det(A) = -8.$$

5. **a)** Is there a real 2×2 matrix A that satisfies $A^4 = -I_2$? Either write such an A, or show that no such A exists.
(hint: think geometrically! The matrix $-I_2$ represents rotation by π radians).

b) Is there a real 3×3 matrix A that satisfies $A^4 = -I_3$? Either write such an A, or show that no such A exists.

Solution.

a) Yes. Just take A to be the matrix of counterclockwise rotation by $\frac{\pi}{4}$ radians:

$$A = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\\1 & 1\end{pmatrix}.$$

Then A^2 gives rotation c.c. by $\dfrac{\pi}{2}$ radians, A^3 gives rotation c.c. by $\dfrac{3\pi}{4}$ radians, and A^4 gives rotation c.c. by π radians, which has matrix $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I_2$.

b) No. If $A^4 = -I$ then

$$[\det(A)]^4 = \det(A^4) = \det(-I) = (-1)^3 = -1.$$

In other words, if $A^4 = -I$ then $[\det(A)]^4 = -1$, which is impossible since $\det(A)$ is a real number.

Similarly, $A^4 = -I$ is impossible if A is 5×5, 7×7, etc.