# Chapter 3
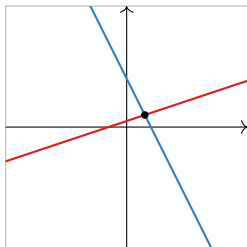
Systems of Linear Equations: Geometry

We want to think about the *algebra* in linear algebra (systems of equations and their solution sets) in terms of *geometry* (points, lines, planes, etc).

$$
\begin{aligned}
x - 3y &= -3 \\
2x + y &= 8
\end{aligned}
$$

⤳

This will give us better insight into the properties of systems of equations and their solution sets.

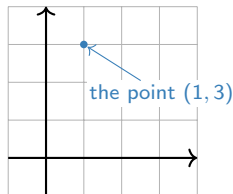Remember: I expect you to be able to draw pictures!

# Section 3.1

Vectors

We have been drawing elements of $\mathbf{R}^n$ as points in the line, plane, space, etc.
We can also draw them as arrows.

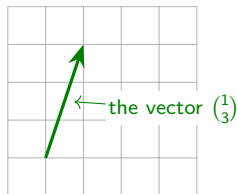### Definition

A **point** is an element of $\mathbf{R}^n$, drawn as a point (a dot).

the point $(1, 3)$

A **vector** is an element of $\mathbf{R}^n$, drawn as an arrow. When we think of an element of $\mathbf{R}^n$ as a vector, we'll usually write it vectically, like a matrix with one column:

$$v = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$
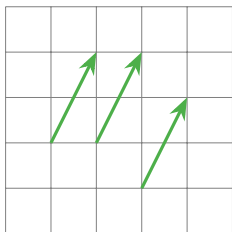
the vector $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$

[interactive]

The difference is purely psychological: *points and vectors are just lists of numbers*.

So why make the distinction?

A vector need not start at the origin: *it can be located anywhere*! In other words, an arrow is determined by its length and its direction, not by its location.



These arrows all represent the vector $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

However, unless otherwise specified, we'll assume a vector starts at the origin.

### Definition

- We can add two vectors together:

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} + \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a + x \\ b + y \\ c + z \end{pmatrix}.$$

- We can multiply, or **scale**, a vector by a real number $c$:

$$c \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} c \cdot x \\ c \cdot y \\ c \cdot z \end{pmatrix}.$$

We call $c$ a **scalar** to distinguish it from a vector. If $v$ is a vector and $c$ is a scalar, $cv$ is called a **scalar multiple** of $v$.

(And likewise for vectors of length $n$.) For instance,

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 5 \\ 7 \\ 9 \end{pmatrix} \quad \text{and} \quad -2 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -2 \\ -4 \\ -6 \end{pmatrix}.$$
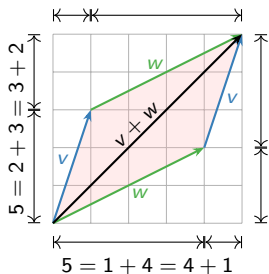
## The parallelogram law for vector addition

Geometrically, the sum of two vectors $v, w$ is obtained as follows: place the tail of $w$ at the head of $v$. Then $v + w$ is the vector whose tail is the tail of $v$ and whose head is the head of $w$. Doing this both ways creates a parallelogram. For example,

$$\begin{pmatrix} 1 \\ 3 \end{pmatrix} + \begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \end{pmatrix}.$$

Why? The width of $v + w$ is the sum of the widths, and likewise with the heights. [interactive]

## Vector subtraction

Geometrically, the difference of two vectors $v, w$ is obtained as follows: place the tail of $v$ and $w$ at the same point. Then $v - w$ is the vector from the head of $w$ to the head of $v$. For example,

$$\begin{pmatrix} 1 \\ 4 \end{pmatrix} - \begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}.$$

Why? If you add $v - w$ to $w$, you get $v$. [interactive]
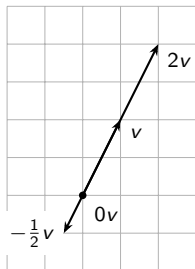
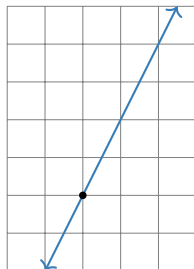This works in higher dimensions too!
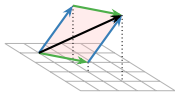
Scalar multiples of a vector

These have the same *direction* but a different *length*.

Some multiples of $v$.



All multiples of $v$.



$$v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$2v = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

$$-\frac{1}{2}v = \begin{pmatrix} -\frac{1}{2} \\ -1 \end{pmatrix}$$

$$0v = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

[interactive]

So the scalar multiples of $v$ form a *line*.

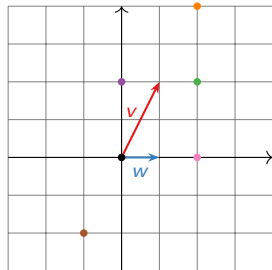We can add and scalar multiply in the same equation:

$$w = c_1 v_1 + c_2 v_2 + \cdots + c_p v_p$$

where $c_1, c_2, \ldots, c_p$ are scalars, $v_1, v_2, \ldots, v_p$ are vectors in $\mathbf{R}^n$, and $w$ is a vector in $\mathbf{R}^n$.

### Definition
We call $w$ a **linear combination** of the vectors $v_1, v_2, \ldots, v_p$. The scalars $c_1, c_2, \ldots, c_p$ are called the **weights** or **coefficients**.

### Example



Let $v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $w = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

What are some linear combinations of $v$ and $w$?
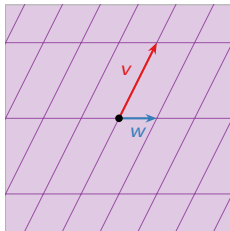
- $v + w$
- $v - w$
- $2v + 0w$
- $2w$
- $-v$

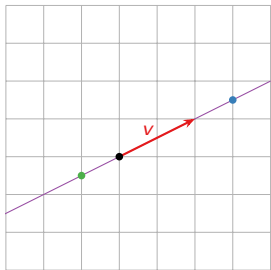[interactive: 2 vectors]     [interactive: 3 vectors]

Poll

Is there any vector in $\mathbf{R}^2$ that is *not* a linear combination of $v$ and $w$?

No: in fact, *every* vector in $\mathbf{R}^2$ is a combination of $v$ and $w$.



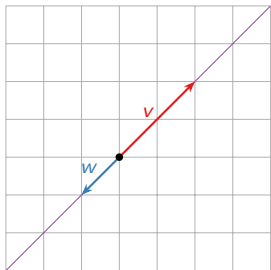(The purple lines are to help measure *how much* of $v$ and $w$ you need to get to a given point.)

What are some linear combinations of $v = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$?

- $\frac{3}{2}v$
- $-\frac{1}{2}v$
- ...

What are *all* linear combinations of $v$?

All vectors $cv$ for $c$ a real number. I.e., all *scalar multiples* of $v$. These form a *line*.



## Question

What are all linear combinations of

$$v = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \quad \text{and} \quad w = \begin{pmatrix} -1 \\ -1 \end{pmatrix}?$$

Answer: The line which contains both vectors.

What's different about this example and the one on the poll?                    [interactive]

# Section 3.2

Vector Equations and Spans

Question

Is $\begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}$ a linear combination of $\begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix}$?

This means: can we solve the equation

$$x \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix} + y \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}$$

where $x$ and $y$ are the unknowns (the coefficients)? Rewrite:

$$\begin{pmatrix} x \\ 2x \\ 6x \end{pmatrix} + \begin{pmatrix} -y \\ -2y \\ -y \end{pmatrix} = \begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix} \qquad \text{or} \qquad \begin{pmatrix} x - y \\ 2x - 2y \\ 6x - y \end{pmatrix} = \begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}.$$

This is just a system of linear equations:

$$\begin{aligned}
x - y &= 8 \\
2x - 2y &= 16 \\
6x - y &= 3.
\end{aligned}$$

$$\begin{aligned} x - y &= 8 \\ 2x - 2y &= 16 \\ 6x - y &= 3 \end{aligned}$$

$\xrightarrow{\text{matrix form}}$

$$\left( \begin{array}{cc|c} 1 & -1 & 8 \\ 2 & -2 & 16 \\ 6 & -1 & 3 \end{array} \right)$$

$\xrightarrow{\text{row reduce}}$

$$\left( \begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & -9 \\ 0 & 0 & 0 \end{array} \right)$$

$\xrightarrow{\text{solution}}$

$$\begin{aligned} x &= -1 \\ y &= -9 \end{aligned}$$

Conclusion:

$$-\begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix} - 9 \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}$$

[interactive] ⟵ (this is the picture of a *consistent* linear system)

What is the relationship between the original vectors and the matrix form of the linear equation? They have the same columns!

Shortcut: You can make the augmented matrix without writing down the system of linear equations first.

## Vector Equations and Linear Equations

**Summary**

The vector equation

$$x_1 v_1 + x_2 v_2 + \cdots + x_p v_p = b,$$

where $v_1, v_2, \ldots, v_p, b$ are vectors in $\mathbf{R}^n$ and $x_1, x_2, \ldots, x_p$ are scalars, has the same solution set as the linear system with augmented matrix

$$\left( \begin{array}{ccccc|c} | & | & & | & | \\ v_1 & v_2 & \cdots & v_p & b \\ | & | & & | & | \end{array} \right),$$

where the $v_i$'s and $b$ are the columns of the matrix.

So we now have (at least) *two* equivalent ways of thinking about linear systems of equations:

1. Augmented matrices.
2. Linear combinations of vectors (vector equations).

The last one is more geometric in nature.

# Span

It is important to know what are *all* linear combinations of a set of vectors $v_1, v_2, \ldots, v_p$ in $\mathbf{R}^n$: it's exactly the collection of all $b$ in $\mathbf{R}^n$ such that the vector equation (in the unknowns $x_1, x_2, \ldots, x_p$)

$$x_1 v_1 + x_2 v_2 + \cdots + x_p v_p = b$$

has a solution (i.e., is consistent).

## Definition

"the set of"     "such that"

Let $v_1, v_2, \ldots, v_p$ be vectors in $\mathbf{R}^n$. The **span** of $v_1, v_2, \ldots, v_p$ is the collection of all linear combinations of $v_1, v_2, \ldots, v_p$, and is denoted $\mathrm{Span}\{v_1, v_2, \ldots, v_p\}$. In symbols:

$$\mathrm{Span}\{v_1, v_2, \ldots, v_p\} = \left\{ x_1 v_1 + x_2 v_2 + \cdots + x_p v_p \mid x_1, x_2, \ldots, x_p \text{ in } \mathbf{R} \right\}.$$

Synonyms: $\mathrm{Span}\{v_1, v_2, \ldots, v_p\}$ is the subset **spanned by** or **generated by** $v_1, v_2, \ldots, v_p$.

This is the first of several definitions in this class that you simply **must learn**. I will give you other ways to think about Span, and ways to draw pictures, but *this is the definition*. Having a vague idea what Span means will not help you solve any exam problems!

Now we have several equivalent ways of making the same statement:

1. A vector $b$ is in the span of $v_1, v_2, \ldots, v_p$.

2. The vector equation

$$x_1 v_1 + x_2 v_2 + \cdots + x_p v_p = b$$

has a solution.

3. The linear system with augmented matrix

$$\left( \begin{array}{cccc|c} | & | & & | & | \\ v_1 & v_2 & \cdots & v_p & b \\ | & | & & | & | \end{array} \right)$$
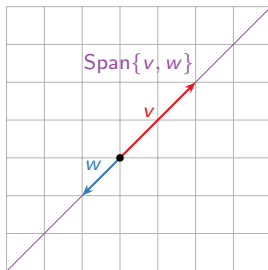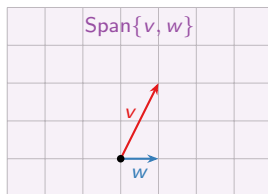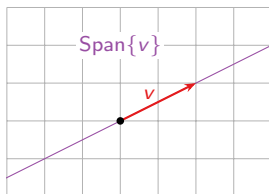
is consistent.

[interactive example] ⟵ (this is the picture of an *inconsistent* linear system)

Note: **equivalent** means that, for any given list of vectors $v_1, v_2, \ldots, v_p, b$, *either* all three statements are true, *or* all three statements are false.
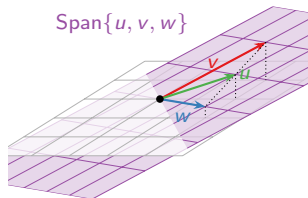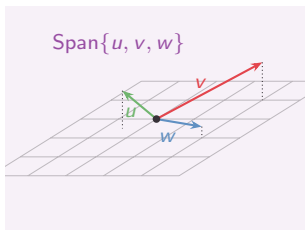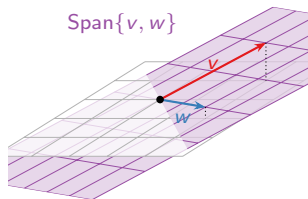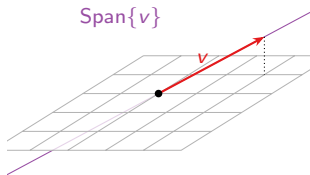
# Pictures of Span

Drawing a picture of Span$\{v_1, v_2, \ldots, v_p\}$ is the same as drawing a picture of all linear combinations of $v_1, v_2, \ldots, v_p$.



[interactive: span of two vectors in $\mathbf{R}^2$]

Span$\{v\}$

Span$\{v, w\}$

Span$\{u, v, w\}$

Span$\{u, v, w\}$

[interactive: span of two vectors in $\mathbf{R}^3$]   [interactive: span of three vectors in $\mathbf{R}^3$]

How many vectors are in Span $\left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$?

A. Zero

B. One

C. Infinity

In general, it appears that $\mathrm{Span}\{v_1, v_2, \ldots, v_p\}$ is the smallest "linear space" (line, plane, etc.) containing the origin and all of the vectors $v_1, v_2, \ldots, v_p$.

We will make this precise later.

The whole lecture was about drawing pictures of systems of linear equations.

- **Points** and **vectors** are two ways of drawing elements of $\mathbf{R}^n$. Vectors are drawn as arrows.
- Vector addition, subtraction, and scalar multiplication have geometric interpretations.
- A **linear combination** is a sum of scalar multiples of vectors. This is also a geometric construction, which leads to lots of pretty pictures.
- The **span** of a set of vectors is the set of all linear combinations of those vectors. It is also fun to draw.
- A system of linear equations is equivalent to a vector equation, where the unknowns are the coefficients of a linear combination.