

Chapter 2

Systems of Linear Equations: Algebra

Section 2.1

Systems of Linear Equations

Recall that \mathbf{R} denotes the collection of all real numbers, i.e. the number line. It contains numbers like $0, -1, \pi, \frac{3}{2}, \dots$

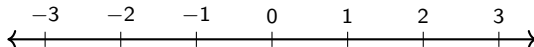
Definition

Let n be a positive whole number. We define

$$\mathbf{R}^n = \text{all ordered } n\text{-tuples of real numbers } (x_1, x_2, x_3, \dots, x_n).$$

Example

When $n = 1$, we just get \mathbf{R} back: $\mathbf{R}^1 = \mathbf{R}$. Geometrically, this is the *number line*.

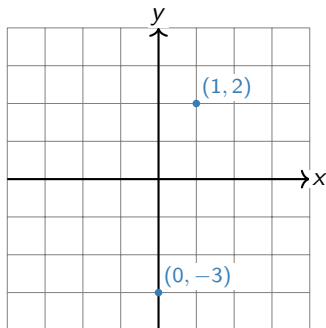


Line, Plane, Space, ...

Continued

Example

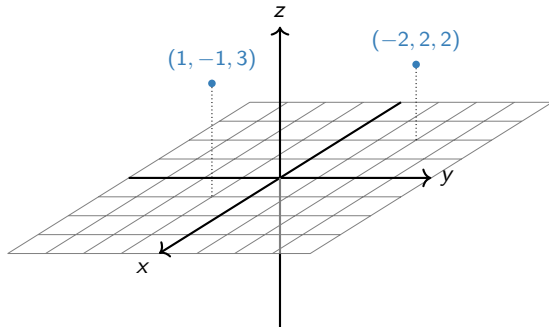
When $n = 2$, we can think of \mathbf{R}^2 as the *plane*. This is because every point on the plane can be represented by an ordered pair of real numbers, namely, its x - and y -coordinates.



We can use the elements of \mathbf{R}^2 to *label* points on the plane, but \mathbf{R}^2 is not defined to be the plane!

Example

When $n = 3$, we can think of \mathbf{R}^3 as the *space* we (appear to) live in. This is because every point in space can be represented by an ordered triple of real numbers, namely, its x -, y -, and z -coordinates.

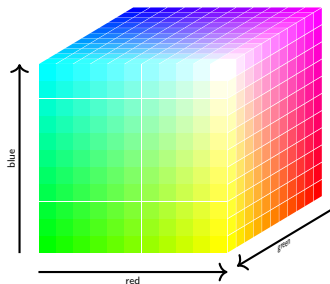


Again, we can use the elements of \mathbf{R}^3 to *label* points in space, but \mathbf{R}^3 is not defined to be space!

Example

All colors you can see can be described by three quantities: the amount of red, green, and blue light in that color. So we could also think of \mathbf{R}^3 as the space of all *colors*:

$$\mathbf{R}^3 = \text{all colors } (r, g, b).$$



Again, we can use the elements of \mathbf{R}^3 to *label* the colors, but \mathbf{R}^3 is not defined to be the space of all colors!

Line, Plane, Space, ...

Continued

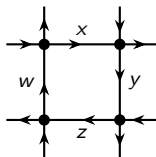
So what is \mathbf{R}^4 ? or \mathbf{R}^5 ? or \mathbf{R}^n ?

...go back to the *definition*: ordered n -tuples of real numbers

$$(x_1, x_2, x_3, \dots, x_n).$$

They're still "geometric" spaces, in the sense that our intuition for \mathbf{R}^2 and \mathbf{R}^3 sometimes extends to \mathbf{R}^n , but they're harder to visualize.

Last time we could have used \mathbf{R}^4 to label the amount of traffic (x, y, z, w) passing through four streets.

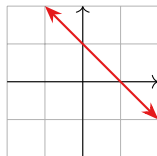


We'll make definitions and state theorems that apply to any \mathbf{R}^n , but we'll only draw pictures for \mathbf{R}^2 and \mathbf{R}^3 .

One Linear Equation

What does the solution set of a linear equation look like?

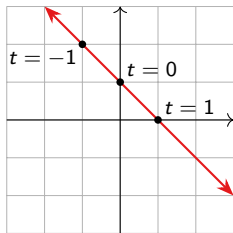
$x + y = 1$ \rightsquigarrow a line in the plane: $y = 1 - x$
This is called the **implicit equation** of the line.



We can write the same line in **parametric form** in \mathbf{R}^2 :

$$(x, y) = (t, 1 - t) \quad t \text{ in } \mathbf{R}.$$

This means that every point on the line has the form $(t, 1 - t)$ for some real number t .



Aside

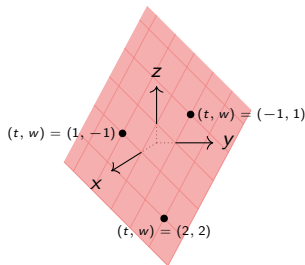
What is a line? A ray that is *straight* and infinite in both directions.

One Linear Equation

Continued

What does the solution set of a linear equation look like?

$x + y + z = 1$ \rightsquigarrow a plane in space:
This is the **implicit equation** of the plane.



[interactive]

Does this plane have a **parametric form**?

$$(x, y, z) = (t, w, 1 - t - w) \quad t, w \text{ in } \mathbf{R}.$$

Note: we are *labeling* the points on the plane by elements (t, w) in \mathbf{R}^2 .

Aside

What is a plane? A flat sheet of paper that's infinite in all directions.

One Linear Equation

Continued

What does the solution set of a linear equation look like?

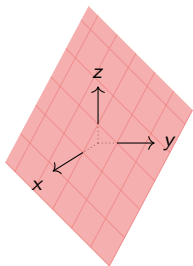
$x + y + z + w = 1 \rightsquigarrow$ a “3-plane” in “4-space”...

[not pictured here]

Is the plane from the previous example equal to \mathbf{R}^2 ?

A. Yes

B. No



No! Every point on this plane is in \mathbf{R}^3 : that means it has three coordinates. For instance, $(1, 0, 0)$. Every point in \mathbf{R}^2 has two coordinates. But we can *label* the points on the plane by \mathbf{R}^2 .

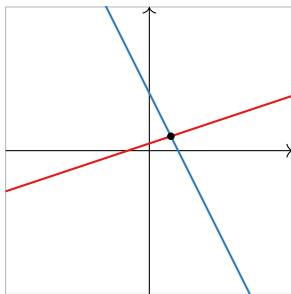
Systems of Linear Equations

What does the solution set of a *system* of more than one linear equation look like?

$$x - 3y = -3$$

$$2x + y = 8$$

... is the *intersection* of two lines, which is a *point* in this case.



In general it's an intersection of lines, planes, etc.

[two planes intersecting]

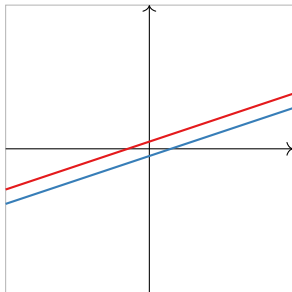
Kinds of Solution Sets

In what other ways can two lines intersect?

$$x - 3y = -3$$

$$x - 3y = 3$$

has no solution: the lines are
parallel.



A system of equations with no solutions is called **inconsistent**.

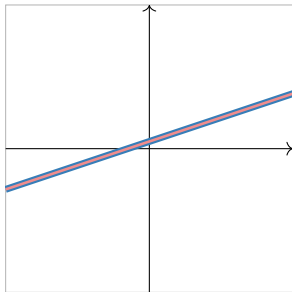
Kinds of Solution Sets

In what other ways can two lines intersect?

$$x - 3y = -3$$

$$2x - 6y = -6$$

has infinitely many solutions:
they are the *same line*.



Note that multiplying an equation by a nonzero number gives the *same solution set*. In other words, they are *equivalent* (systems of) equations.

Summary

- ▶ \mathbf{R}^n is the set of ordered lists of n numbers.
- ▶ \mathbf{R}^n can be used to label geometric objects, like \mathbf{R}^2 can label points in the plane.
- ▶ The solutions of a system equations look like an intersection of lines, planes, etc.
- ▶ Finding all the solutions means finding a **parametric form** of the system of equations.