# MATH 1553
# SAMPLE FINAL EXAM, SPRING 2019

| **Name** | |
|---|---|

Please **read all instructions** carefully before beginning.

- Each problem is worth 10 points. The maximum score on this exam is 100 points.
- You have 170 minutes to complete this exam.
- You may not use any calculators or aids of any kind (notes, text, etc.).
- Unless a problem specifies that no work is required, show your work or you may receive little or no credit, even if your answer is correct.
- If you run out of room on a page, you may use its back side to finish the problem, but please indicate this.
- You may cite any theorem proved in class or in the sections we covered in the text.
- Check your answers if you have time left! Most linear algebra computations can be easily verified for correctness. Good luck!

This is a practice exam. It is meant to be roughly similar in format, length, and difficulty to the real exam. It is not meant as a comprehensive list of study problems.

Please read and sign the following statement.

*I, the undersigned, hereby affirm that I will not share the contents of this exam with anyone. Furthermore, I have not received inappropriate assistance in the midst of nor prior to taking this exam.*

_____

## Problem 1. [1 point each]

True or false. Circle **T** if the statement is *always* true. Otherwise, answer **F**. You do not need to justify your answer. In every case, $A$ is a matrix whose entries are real numbers.

a) **T** **F** Suppose $\{v_1, \ldots, v_6\}$ is a set of vectors that spans $\mathbb{R}^5$. Then $\{v_1, \ldots, v_6\}$ is a basis for $\mathbb{R}^5$.

b) **T** **F** Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a one-to-one linear transformation. Then $n \leq m$.

c) **T** **F** Suppose $A$ is an $n \times n$ matrix and the sum of the columns of $A$ is the zero vector. Then $A$ is not invertible.

d) **T** **F** Suppose $A$ is a square matrix that is diagonalizable and invertible. Then $A^{-1}$ is diagonalizable.

e) **T** **F** Suppose $A$ is a $3 \times 3$ matrix with characteristic polynomial $-\lambda^3 - \lambda^2 - \lambda - 1$. Then $A$ is invertible.

f) **T** **F** Suppose $A$ is a $3 \times 3$ matrix whose characteristic polynomial is $-\lambda^3 - \lambda^2$ and whose null space is a line. Then $A$ is diagonalizable.

g) **T** **F** There is a $2 \times 2$ matrix $A$ so that the solution set of $Ax = \binom{1}{2}$ is the line $y = 2x + 1$ and the solution set of $Ax = \binom{-4}{1}$ is the line $y = 3x - 1$.

h) **T** **F** If $A$ is the standard matrix for an orthogonal projection onto a subspace, then

$$(\mathrm{Nul}A)^{\perp} = \mathrm{Col}A.$$

i) **T** **F** Let $T$ be the linear transformation given by orthogonal projection onto the subspace

$$W = \left\{(x, y, z, w) \text{ in } \mathbf{R}^4 \mid x + y + z + 2w = 0\right\}.$$

Then the dimension of the range of $T$ is 3.

j) **T** **F** There is a stochastic $2 \times 2$ matrix $A$ that has $1 + i$ as an eigenvalue.

## Problem 2. [2 points each]

Short answer. You do not need to show your work, and there is no partial credit. In each case, $A$ is a matrix whose entries are real numbers.

**a)** Suppose that $A$ is a $2 \times 2$ matrix, that 5 is an eigenvalue of $A$, and that $A$ is not diagonalizable. What is the characteristic polynomial of $A$?

**b)** Find a $2 \times 2$ matrix whose column space is the line $y = 2x$ and whose null space is the $x$-axis.

**c)** Suppose $u$ and $v$ are orthogonal vectors with $||u|| = 2$ and $||v|| = 3$.
Compute the dot product $(4u + 5v) \cdot v$.

**d)** Let $A$ be a $3 \times 3$ matrix whose 3-eigenspace is a two-dimensional plane and whose 1-eigenspace is a line. What is the determinant of $A$?

**e)** Suppose that $\det\begin{pmatrix} a & b & c \\ 1 & 2 & 3 \\ 0 & -1 & 5 \end{pmatrix} = 2$. Find $\det\begin{pmatrix} 1 & 2 & 3 \\ 5a+1 & 5b+2 & 5c+3 \\ 1 & 1 & 8 \end{pmatrix}$.

## Problem 3. [(a), (b), (c), (d) are worth 1, 2, 3, 4 points respectively]

Short answer. You do not need to show your work, and there is no partial credit.

**a)** Let $T : \mathbf{R}^3 \to \mathbf{R}^3$ be a linear transformation and suppose $\begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ are in the range of $T$. Write another nonzero vector in the range of $T$ here: $\begin{pmatrix} \quad \\ \quad \\ \quad \end{pmatrix}$

**b)** Suppose that $A$ is a $12 \times 9$ matrix and the solution set to $Ax = 0$ has dimension 7.

(i) Fill in the blank: the dimension of the column space of $A$ is _____.

(ii) Fill in the blank: the dimension of the row space of $A$ is _____.

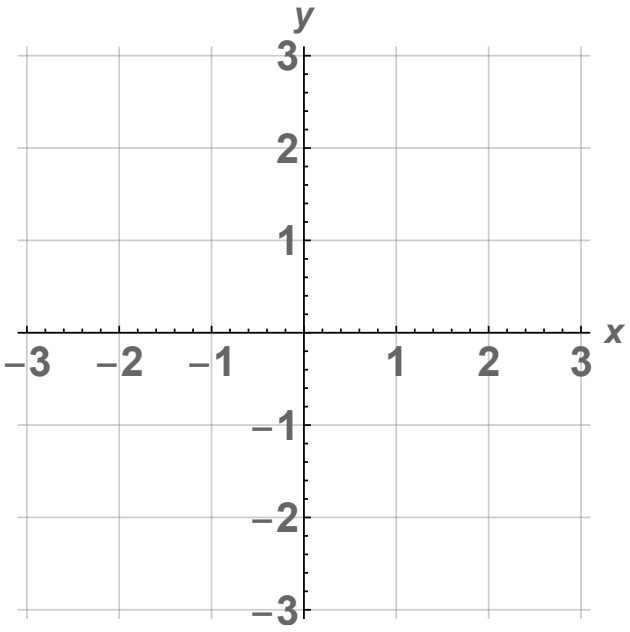
**c)** Suppose $A$ is a stochastic matrix. Which of the following must be true? Circle all that apply.

(i) The sum of entries in each row of $A$ is equal to 1.

(ii) The sum of entries in each column of $A$ is equal to 1.

(iii) No entry of $A$ is greater than 1.

**d)** Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the transformation of reflection across the line $y = 3x$, and let $A$ be the standard matrix for $T$. Draw each eigenspace of $A$ precisely, and clearly label each eigenspace with its eigenvalue.



## Problem 4.

[2 points for (a); 4 points each for (b) and (c)]

No work is necessary in parts (a) and (b). Show your work in part (c).

**a)** Complete the following definition.
A vector $v$ in $\mathbf{R}^n$ is an *eigenvector* of an $n \times n$ matrix $A$ if ...

**b)** Let $T : \mathbf{R}^n \to \mathbf{R}^m$ be a linear transformation with standard matrix $A$. Which of the following conditions guarantee that $T$ *must* be one-to-one? Circle all that apply.

(i) $A$ has $m$ pivots.

(ii) The columns of $A$ are linearly independent.

(iii) For each input vector $x$ in $\mathbf{R}^n$, there is exactly one output vector $T(x)$ in $\mathbf{R}^m$.

(iv) The equation $Ax = b$ has exactly one solution for each $b$ in $\mathbf{R}^m$.

**c)** The inverse of $A = \begin{pmatrix} 1 & 3 & 5 \\ -1 & -4 & -8 \\ 1 & 5 & 12 \end{pmatrix}$ is $A^{-1} = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$.

Free response. For all problems remaining, show all work, and justify your answers where appropriate. A correct answer without proper work may receive little or no credit.

# Problem 5.

Parts (a) and (b) are unrelated.

**a)** [6 points] Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation obtained by reflecting over the line $y = 0$ and then rotating by 45° counterclockwise.

(i) Find the standard matrix for $T$. Write your answer here: $\begin{pmatrix} & \\ & \end{pmatrix}$

(ii) Is $T$ one-to-one? Justify your answer.

**b)** [4 points] Define linear transformations $S : \mathbf{R}^3 \to \mathbf{R}^2$ and $U : \mathbf{R}^2 \to \mathbf{R}^3$ by

$$S(x_1, x_2, x_3) = (x_1 - x_2 + 2x_3, x_3) \quad \text{and} \quad U(x_1, x_2) = (x_1 + x_2, 3x_1 - x_2, x_1).$$

Find the standard matrix for $S \circ U$. Write your answer here: $\begin{pmatrix} & \\ & \\ & \end{pmatrix}$

## Problem 6.

Your roommate Karxon has given you the following matrix $A$ and its reduced row echelon form:

$$A = \begin{pmatrix} 1 & 2 & -1 & -1 \\ -1 & -2 & 2 & 5 \\ 2 & 4 & 0 & 6 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

**a)** [4 points] Find a basis for Nul $A$.

**b)** [6 points] Find the closest vector $w$ to $\begin{pmatrix} -3 \\ -2 \\ 5 \end{pmatrix}$ in Col $A$. $\quad w = \begin{pmatrix} \quad \\ \quad \\ \quad \end{pmatrix}$

## Problem 7.

Let $A = \begin{pmatrix} 3 & 5 & 2 \\ 0 & 2 & 0 \\ 1 & -2 & 4 \end{pmatrix}$. Its eigenvalues are $\lambda = 2$ and $\lambda = 5$.

**a)** [5 points] Find a basis for each eigenspace of $A$. Enter your answers below.

Basis for 2-eigenspace: $\left\{ \quad \right\}$ Basis for 5-eigenspace: $\left\{ \quad \right\}$.

**b)** [3 points] Is $A$ diagonalizable? If your answer is yes, write an invertible matrix $C$ and diagonal matrix $D$ so that $A = CDC^{-1}$. If your answer is no, justify why $A$ is not diagonalizable.

**c)** [2 points] Find a basis for $(\text{Nul}A)^{\perp}$. Write your answer here: $\left\{ \quad \right\}$.

## Problem 8.

Let $A = \begin{pmatrix} -1 & -2 \\ 5 & 5 \end{pmatrix}$.

**a)** [6 points] Find the characteristic polynomial of $A$ and the eigenvalues of $A$. Write your answers for the eigenvalues in the spaces below.
The eigenvalue with *positive* imaginary part is $\lambda_1 =$ _____.

The eigenvalue with *negative* imaginary part is $\lambda_2 =$ _____.

**b)** [4 points] For each eigenvalue of $A$, find a corresponding eigenvector.
Write your answers below:

An eigenvector for $\lambda_1$ (the eigenvalue with *positive* imaginary part) is $v_1 = \begin{pmatrix} \quad \\ \quad \end{pmatrix}$.

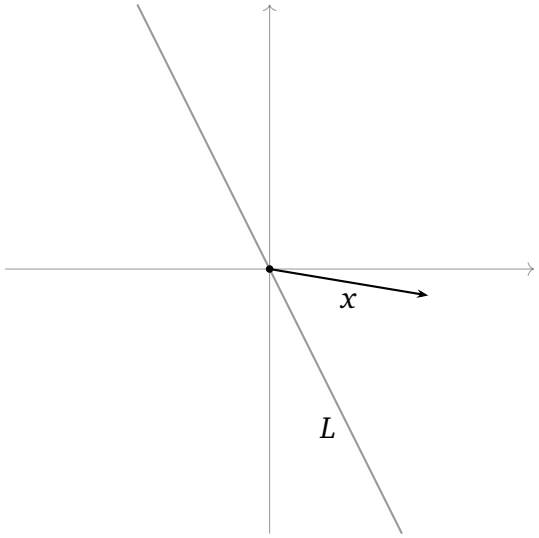
An eigenvector for $\lambda_2$ (the eigenvalue with *negative* imaginary part) is $v_2 = \begin{pmatrix} \quad \\ \quad \end{pmatrix}$.

# Problem 9.

Let $L$ be the line in $\mathbf{R}^2$ spanned by $u = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

Recall our notation: if $x$ is a vector, then $x_L$ is the orthogonal projection of $x$ onto $L$.

**a)** [3 points] Let $x$ be the vector graphed below.
Carefully sketch three things: $L^\perp$, $x_L$, and $x_{L^\perp}$. Clearly label each.




**b)** [4 points] Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation given by orthogonal projection onto $L$. Find the standard matrix for $T$.

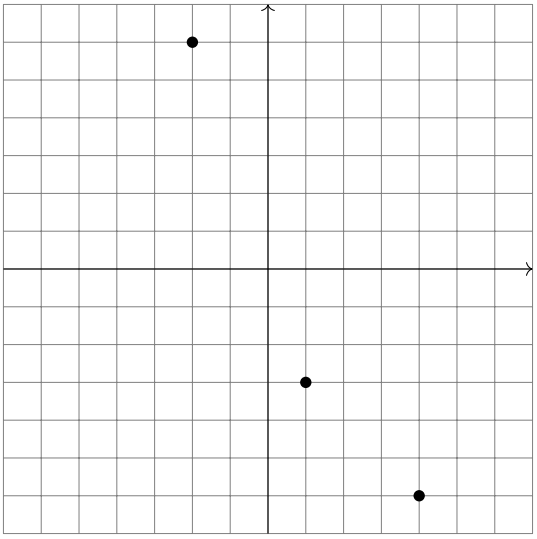
**c)** [3 pts] Compute $y_L$ and $y_{L^\perp}$ for $y = \begin{pmatrix} 5 \\ 5 \end{pmatrix}$. $\quad y_L = \begin{pmatrix} \quad \\ \quad \end{pmatrix} \quad y_{L^\perp} = \begin{pmatrix} \quad \\ \quad \end{pmatrix}$

## Problem 10.

Consider the data points $(-2, 6)$, $(1, -3)$, and $(4, -6)$. Find the best-fit line for these data points. Enter your answer in the space below.

$y =$ _____$x$ + _____.

For your benefit, the data points are plotted at the bottom of the page, so that you may check your answer by plotting your line to make sure it looks reasonable (the graph will not be graded; it is there solely for you to check your work).



**Scrap paper. This page will not be graded under any circumstances**