MATH 1553
SAMPLE FINAL EXAM, FALL 2019

Name	

Please **read all instructions** carefully before beginning.

- Each problem is worth 10 points. The maximum score on this exam is 100 points.
- You have 170 minutes to complete this exam.
- You may not use any calculators or aids of any kind (notes, text, etc.).
- Unless a problem specifies that no work is required, show your work or you may receive little or no credit, even if your answer is correct.
- If you run out of room on a page, you may use its back side to finish the problem, but please indicate this.
- You may cite any theorem proved in class or in the sections we covered in the text.
- Check your answers if you have time left! Most linear algebra computations can be easily verified for correctness. Good luck!

This is a practice exam. It is meant to be roughly similar in format, length, and difficulty to the real exam. It is not meant as a comprehensive list of study problems.

Please read and sign the following statement.

I, the undersigned, hereby affirm that I will not share the contents of this exam with anyone. Furthermore, I have not received inappropriate assistance in the midst of nor prior to taking this exam.

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Problem 1. [1 point each]

TRUE or FALSE. Circle **T** if the statement is *always* true. Otherwise, answer **F**. You do not need to show work or justify your answer.

a) **T** **F** If $T : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation that satisfies

$$T\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\2\\0\end{pmatrix}, \qquad T\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}1\\0\\-1\end{pmatrix}, \qquad T\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}3\\0\\0\end{pmatrix},$$

then T is one to one.

b) **T** **F** If the system $Ax = \begin{pmatrix}1\\-2\end{pmatrix}$ has a unique solution $\begin{pmatrix}3\\1\end{pmatrix}$, then the homogeneous equation $Ax = 0$ has only the trivial solution.

c) **T** **F** If A and B are $n \times n$ matrices then $\det(A+B) = \det(A) + \det(B)$.

d) **T** **F** If A is a 5×7 and $\dim(\text{Nul } A) = 4$, then $\dim(\text{Row } A) = 3$.

e) **T** **F** The set $W = \left\{ \begin{pmatrix}x\\y\\z\\w\end{pmatrix} \text{ in } \mathbf{R}^4 \;\middle|\; x - y = z - w \right\}$ is a 3-dimensional subspace of $\mathbf{R}^4$.

f) **T** **F** If A is a 3×3 matrix with characteristic polynomial

$$\det(A - \lambda I) = -\lambda(2-\lambda)(3-\lambda),$$

then A is diagonalizable.

g) **T** **F** Suppose W is a subspace of $\mathbf{R}^n$ and B is the matrix for orthogonal projection onto W. Then for every x in $\mathbf{R}^n$, we have $Bx = x$ or $Bx = 0$.

h) **T** **F** Each inconsistent system $Ax = b$ has exactly one least squares solution.

i) **T** **F** Any $n \times n$ matrix with n linearly independent eigenvectors in $\mathbf{R}^n$ is diagonalizable.

j) **T** **F** The vector $\begin{pmatrix}0.6\\0.4\end{pmatrix}$ is the steady state vector of the matrix $\begin{pmatrix}0.4 & 0.6\\0.6 & 0.4\end{pmatrix}$.

Problem 2.

Short answer. You do not need to show your work on (a) or (b), but briefly show your work on part (c).

a) Let $\{v_1, v_2, \ldots, v_n\}$ be vectors in $\mathbb{R}^m$. Which of the following conditions imply that these vectors are linearly independent? Circle all that apply.

(i) The vector equation $x_1v_1 + x_2v_2 + \cdots + x_nv_n = 0$ has a unique solution.

(ii) The subspace $\mathrm{Span}\{v_1, v_2, \ldots, v_n\}$ has dimension n.

(iii) The RREF of the matrix $A = \begin{pmatrix} | & | & \cdots & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & \cdots & | \end{pmatrix}$ has a pivot in every column.

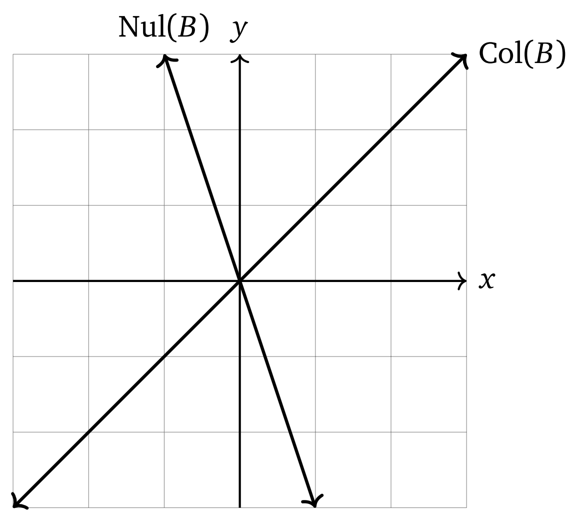
b) Let A be a 3×4 matrix. Which of the following statements *can* be true? Circle all that apply.

(i) The transformation $T : \mathbb{R}^4 \to \mathbb{R}^3$ defined by $T(x) = Ax$ is one to one.

(ii) The rank of A is equal to 2 and $\mathrm{Nul}(A)$ is the x-axis.

(iii) The column space of A and the null space of A have the same dimension.

c) The null space and column space of another matrix B are given in the picture. Find such a matrix B.



Problem 3.

Short answer. On (a) and (b), you do not need to show your work, and there is no partial credit. Show your work in (c).

a) Suppose A is an $m \times n$ matrix and the only solution to the homogeneous equation $Ax = 0$ is the trivial solution $x = 0$. Let T be the matrix transformation $T(x) = Ax$. Which of the following *must* be true? Circle all that apply.

(i) T is onto.

(ii) T is one-to-one.

(iii) If $m = n$, then A is invertible.

(iv) If $m > n$, then the equation $Ax = b$ is inconsistent for at least one b in $\mathbf{R}^m$.

b) The equation $\begin{pmatrix} 1 & 0 \\ 0 & -1 \\ 3 & 0 \end{pmatrix} x = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ has least-squares solution $\widehat{x} = \begin{pmatrix} 2/5 \\ -2 \end{pmatrix}$. What is the closest vector to $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ in Span $\left\{ \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \right\}$? Enter your answer here: $\begin{pmatrix} \\ \\ \\ \end{pmatrix}$

c) Let W be the set of all vectors in $\mathbf{R}^3$ of the form (a, b, a) where a and b are real numbers. Find a basis for $W^{\perp}$.

Problem 4.

Short answer. Assume that the entries in all matrices are real numbers. You do not need to show your work or justify your answers.

a) Give an example of a 3×3 matrix with characteristic polynomial $(3-\lambda)(2-\lambda)^2$.

b) Give an example of a 3×3 matrix with eigenvector $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

c) Give an example of a 2×2 matrix that has no real eigenvalues.

d) Give an example of a 3×3 matrix A with exactly one eigenvalue $\lambda = 2$, so that the 2-eigenspace of A is a line.

Problem 5.

a) Find the matrix of the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ which rotates by $90°$ counterclockwise. Enter your answer here: $\begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix}$

b) Find the matrix of the transformation defined by $U\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x+y \\ 3y-x \\ x-3y \end{pmatrix}$.

Enter your answer here: $\begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix}$

c) Circle which transformation makes sense: $T \circ U \qquad U \circ T$

Find the standard matrix for the transformation you circled, and enter it below.

$$\begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix}$$

d) Find A^{-1} if $A = \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}$.

Problem 6.

Consider the matrix A below and its reduced row echelon form:

$$A = \begin{pmatrix} 1 & 2 & -1 & -2 \\ 0 & -3 & 3 & 0 \\ -2 & 2 & -4 & 4 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & 0 & 1 & -2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

a) Write a basis for Col(A).

b) Find a basis for Nul(A).

c) Fill in the blank: rank(A) $=$ ________________.

d) Is there a matrix B such that $\text{Nul}(B) = \text{Col}(A)$? If so, find such a matrix B. If not, show that no such B exists.

Problem 7.

Parts (a), (b), (c), and (d) are unrelated.

a) Suppose that v and w are eigenvectors of a matrix A corresponding to the eigenvalues 4 and -1, respectively. Find $A(2v+3w)$ in terms of v and w.

b) Suppose that for two 5×5 matrices A and B, we have

$$\det(A) = 3, \quad \det(A^{-1}B) = 7.$$

Find $\det(B)$.

c) Suppose that A is a positive stochastic matrix with steady-state vector $\begin{pmatrix} 3/10 \\ 7/10 \end{pmatrix}$. What vector does $A^n \begin{pmatrix} 350 \\ 50 \end{pmatrix}$ approach as n becomes very large? Fully simplify your answer.

d) Compute the orthogonal projection of $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ onto $\mathrm{Span}\left\{\begin{pmatrix} 2 \\ -3 \end{pmatrix}\right\}$.

Problem 8.

Consider the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 4 & 4 \\ 1 & 1 & 1 \end{pmatrix}$$

a) Find the characteristic polynomial and the eigenvalues of A.

b) For each eigenvalue of A, find one corresponding eigenvector.

c) Find an invertible 3×3 matrix C and a diagonal matrix D so that $A = CDC^{-1}$.

Problem 9.

Let $W = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \text{ in } \mathbf{R}^3 \;\middle|\; x_1 + x_2 + x_3 = 0 \right\}$ and $x = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$.

(i) Find a basis for W.

(ii) Find x_W, the orthogonal projection of x onto W.

(iii) Find $x_{W^\perp}$

Problem 10.

Use least squares to find the best-fit line $y = Mx + B$ for the data points

$$(0,0), \qquad (1,2), \qquad (3,-1).$$

Enter your answer below:

$$y = \underline{\qquad\quad} x \ + \ \underline{\qquad\quad}.$$

You must show appropriate work. If you simply guess a line or estimate the equation for the line based on the data points, you will receive little or no credit, even if your answer is correct or nearly correct.

Scrap paper. This page will not be graded under any circumstances