MATH 1553
SAMPLE FINAL EXAM, FALL 2019

Name	

Please **read all instructions** carefully before beginning.

- Each problem is worth 10 points. The maximum score on this exam is 100 points.
- You have 170 minutes to complete this exam.
- You may not use any calculators or aids of any kind (notes, text, etc.).
- Unless a problem specifies that no work is required, show your work or you may receive little or no credit, even if your answer is correct.
- If you run out of room on a page, you may use its back side to finish the problem, but please indicate this.
- You may cite any theorem proved in class or in the sections we covered in the text.
- Check your answers if you have time left! Most linear algebra computations can be easily verified for correctness. Good luck!

This is a practice exam. It is meant to be roughly similar in format, length, and difficulty to the real exam. It is not meant as a comprehensive list of study problems.

Please read and sign the following statement.

I, the undersigned, hereby affirm that I will not share the contents of this exam with anyone. Furthermore, I have not received inappropriate assistance in the midst of nor prior to taking this exam.

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Problem 1.

[1 point each]

TRUE or FALSE. Circle **T** if the statement is *always* true. Otherwise, answer **F**. You do not need to show work or justify your answer.

a) **T** **F** If $T : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation that satisfies

$$T\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\2\\0\end{pmatrix}, \qquad T\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}1\\0\\-1\end{pmatrix}, \qquad T\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}3\\0\\0\end{pmatrix},$$

then T is one to one.

b) **T** **F** If the system $Ax = \begin{pmatrix}1\\-2\end{pmatrix}$ has a unique solution $\begin{pmatrix}3\\1\end{pmatrix}$, then the homogeneous equation $Ax = 0$ has only the trivial solution.

c) **T** **F** If A and B are $n \times n$ matrices then $\det(A+B) = \det(A) + \det(B)$.

d) **T** **F** If A is a 5×7 and $\dim(\mathrm{Nul}\ A) = 4$, then $\dim(\mathrm{Row}\ A) = 3$.

e) **T** **F** The set $W = \left\{ \begin{pmatrix}x\\y\\z\\w\end{pmatrix} \text{ in } \mathbf{R}^4 \;\middle|\; x - y = z - w \right\}$ is a 3-dimensional subspace of $\mathbf{R}^4$.

f) **T** **F** If A is a 3×3 matrix with characteristic polynomial

$$\det(A - \lambda I) = -\lambda(2-\lambda)(3-\lambda),$$

then A is diagonalizable.

g) **T** **F** Suppose W is a subspace of $\mathbf{R}^n$ and B is the matrix for orthogonal projection onto W. Then for every x in $\mathbf{R}^n$, we have $Bx = x$ or $Bx = 0$.

h) **T** **F** Each inconsistent system $Ax = b$ has exactly one least squares solution.

i) **T** **F** Any $n \times n$ matrix with n linearly independent eigenvectors in $\mathbf{R}^n$ is diagonalizable.

j) **T** **F** The vector $\begin{pmatrix}0.6\\0.4\end{pmatrix}$ is the steady state vector of the matrix $\begin{pmatrix}0.4 & 0.6\\0.6 & 0.4\end{pmatrix}$.

Solution to problem 1.

a) True. We can see this in many ways: the matrix $A = \begin{pmatrix} 0 & 1 & 3 \\ 2 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$ has linearly independent columns / a pivot in each column / nonzero determinant / columns that clearly span $\mathbf{R}^3$.

b) True. The solution set to any consistent system $Ax = b$ is a translation of the solution set to $Ax = 0$. Since the $Ax = b$ solution set is unique, so is the solution set to $Ax = 0$.

c) False. For example,

$$\det\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1 \qquad \text{but} \qquad \det\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \det\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = 0 + 0 = 0.$$

d) True. Since $\dim(\mathrm{Nul}(A)) = 4$ and $\mathrm{Nul}A$ is a subspace of $\mathbf{R}^7$, it follows that

$$\dim((\mathrm{Nul}\ A)^{\perp}) = 3.$$

Since $(\mathrm{Nul}\ A)^{\perp} = \mathrm{Row}\ A$ this means $\dim(\mathrm{Row}\ A) = 3$.

e) True. Note $W = \mathrm{Nul}\begin{pmatrix} 1 & -1 & -1 & 1 \end{pmatrix}$ so it is a subspace of $\mathbf{R}^4$, and the system $\left(\begin{array}{cccc|c} 1 & -1 & -1 & 1 & 0 \end{array}\right)$ has three free variables so $\dim(W) = 3$.

f) True: A is a 3×3 matrix with 3 distinct real eigenvalues, so A is diagonalizable.

g) False: For example, if W is projection onto the x-axis in $\mathbf{R}^3$ then $B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, and $B\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ is neither $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ nor $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, since

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

h) False: the condition for uniqueness is that A have linearly independent columns, not that the system be inconsistent. For example, if $A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{pmatrix}$ and $b = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ then $Ax = b$ is inconsistent but there are infinitely many least-squares solutions: $\left(A^T A \mid A^T b\right) = \left(\begin{array}{cc|c} 3 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right)$ which has infinitely many solutions.

i) True, the n linearly independent vectors in $\mathbf{R}^n$ must form a basis for $\mathbf{R}^n$.

j) False. The steady-state vector of the matrix is $\begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$.

Problem 2.

Short answer. You do not need to show your work on (a) or (b), but briefly show your work on part (c).

a) Let $\{v_1, v_2, \ldots, v_n\}$ be vectors in $\mathbb{R}^m$. Which of the following conditions imply that these vectors are linearly independent? Circle all that apply.

$\boxed{\text{(i) The vector equation } x_1v_1 + x_2v_2 + \cdots + x_nv_n = 0 \text{ has a unique solution.}}$

$\boxed{\text{(ii) The subspace Span}\{v_1, v_2, \ldots, v_n\} \text{ has dimension } n}$.

$\boxed{\text{(iii) The RREF of the matrix } A = \begin{pmatrix} | & | & \cdots & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & \cdots & | \end{pmatrix} \text{ has a pivot in every column.}}$

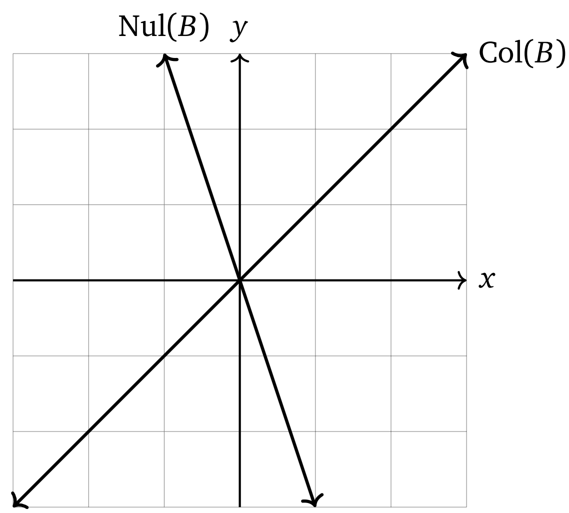
b) Let A be a 3×4 matrix. Which of the following statements *can* be true? Circle all that apply.

(i) The transformation $T : \mathbb{R}^4 \to \mathbb{R}^3$ defined by $T(x) = Ax$ is one to one.

(ii) The rank of A is equal to 2 and Nul(A) is the x-axis.

$\boxed{\text{(iii) The column space of } A \text{ and the null space of } A \text{ have the same dimension.}}$

c) The null space and column space of another matrix B are given in the picture. Find such a matrix B.



We need $\mathrm{Nul}(B) = \mathrm{Span}\{\begin{pmatrix} 1 \\ -3 \end{pmatrix}\}$ and $\mathrm{Col}(B) = \mathrm{Span}\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\}$. Many answers possible, for example $B = \begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix}$

Problem 3.

Short answer. On (a) and (b), you do not need to show your work, and there is no partial credit. Show your work in (c).

a) Suppose A is an $m \times n$ matrix and the only solution to the homogeneous equation $Ax = 0$ is the trivial solution $x = 0$. Let T be the matrix transformation $T(x) = Ax$. Which of the following *must* be true? Circle all that apply.

(i) T is onto.

$\boxed{\text{(ii) } T \text{ is one-to-one.}}$

$\boxed{\text{(iii) If } m = n, \text{ then } A \text{ is invertible.}}$

$\boxed{\text{(iv) If } m > n, \text{ then the equation } Ax = b \text{ is inconsistent for at least one } b \text{ in } \mathbf{R}^m.}$

b) The equation $\begin{pmatrix} 1 & 0 \\ 0 & -1 \\ 3 & 0 \end{pmatrix} x = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ has least-squares solution $\widehat{x} = \begin{pmatrix} 2/5 \\ -2 \end{pmatrix}$. What is the closest vector to $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ in $\text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \right\}$? Enter your answer here: $\boxed{\begin{pmatrix} 2/5 \\ 2 \\ 6/5 \end{pmatrix}}$.

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \\ 3 & 0 \end{pmatrix} \widehat{x} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \\ 3 & 0 \end{pmatrix} \begin{pmatrix} 2/5 \\ -2 \end{pmatrix} = \begin{pmatrix} 2/5 \\ 2 \\ 6/5 \end{pmatrix}$$

c) Let W be the set of all vectors in $\mathbf{R}^3$ of the form (a, b, a) where a and b are real numbers. Find a basis for $W^\perp$.

We see $W = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$, so $W^\perp = \text{Nul}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$.

Thus $x_1 = -x_3$, $x_2 = 0$, and x_3 is free, so a basis for $W^\perp$ is $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$.

Problem 4.

Short answer. Assume that the entries in all matrices are real numbers. You do not need to show your work or justify your answers.

a) Give an example of a 3×3 matrix with characteristic polynomial $(3-\lambda)(2-\lambda)^2$.

Many examples possible, for example $A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

b) Give an example of a 3×3 matrix with eigenvector $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

Many examples possible, for example $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

c) Give an example of a 2×2 matrix that has no real eigenvalues.

Many examples possible. Any rotation matrix aside of I and $-I$ will work, for example $A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ or $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

d) Give an example of a 3×3 matrix A with exactly one eigenvalue $\lambda = 2$, so that the 2-eigenspace of A is a line.

We need only one eigenvalue $\lambda = 2$, and we need $A - 2I$ to have two pivots so that we will only have one free variable in the homogeneous equation $(A-2I)x = 0$. Some examples are

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}, \qquad A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}, \qquad A = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 2 \end{pmatrix}.$$

Some NON-examples are $\begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$ and $\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$, which have the wrong dimension for the 2-eigenspace.

Problem 5.

a) Find the matrix of the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ which rotates by $90°$ counterclockwise.

b) Find the matrix of the transformation defined by $U\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x+y \\ 3y-x \\ x-3y \end{pmatrix}$.

c) Circle which transformation makes sense: $T \circ U \quad U \circ T$

Find the standard matrix for the transformation you circled, and enter it below.

d) Find A^{-1} if $A = \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}$.

Solution.

a) $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

b) $B = \begin{pmatrix} 2 & 1 \\ -1 & 3 \\ 1 & -3 \end{pmatrix}$.

c) $U \circ T$ makes sense, and $BA = \begin{pmatrix} 2 & 1 \\ -1 & 3 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 3 & 1 \\ -3 & -1 \end{pmatrix}$

d) $A^{-1} = \dfrac{1}{2(1)-(-1)(1)}\begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix} = \dfrac{1}{3}\begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1/3 & -1/3 \\ 1/3 & 2/3 \end{pmatrix}$.

Problem 6.

Consider the matrix A below and its reduced row echelon form:

$$A = \begin{pmatrix} 1 & 2 & -1 & -2 \\ 0 & -3 & 3 & 0 \\ -2 & 2 & -4 & 4 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & 0 & 1 & -2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

a) Write a basis for Col(A).

We use the pivot columns. In fact, any pair of columns of A will work here except the 1st and 4th (which are scalar multiples of each other).

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix} \right\}.$$

b) Find a basis for Nul(A).

From the RREF of A we see the homogeneous solution gives

$$x_1 = -x_3 + 2x_4, \qquad x_2 = x_3, \qquad x_3 = x_3 \text{ (free)} \qquad x_4 = x_4 \text{ (free)},$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -x_3 + 2x_4 \\ x_3 \\ x_3 \\ 0 \end{pmatrix} = x_3 \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 2 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \qquad \text{Basis} : \left\{ \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

c) Fill in the blank: rank(A) $=$ 2.

d) Is there a matrix B such that Nul(B) $=$ Col(A)? If so, find such a matrix B. If not, show that no such B exists.

Let $W = \text{Col}(A)$, so $W = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix} \right\}.$

We want B so that $\text{Nul}\, B = W$. From the identity $\text{Nul}\, B = (\text{Row}\, B)^{\perp}$ this means

$$(\text{Row}\, B)^{\perp} = W \qquad \text{so} \qquad \text{Row}\, B = W^{\perp}.$$

For $W^{\perp}$ we solve $\left(\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 2 & -3 & 2 & 0 \end{array}\right)$ which gives $\left\{ \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \right\}$ as a basis for $W^{\perp}$.

We can choose $B = \begin{pmatrix} 2 & 2 & 1 \end{pmatrix}$ or any rank one matrix whose rows are constant multiples of $\begin{pmatrix} 2 & 2 & 1 \end{pmatrix}$.

Problem 7.

Parts (a), (b), (c), and (d) are unrelated.

a) Suppose that v and w are eigenvectors of a matrix A corresponding to the eigenvalues 4 and -1, respectively. Find $A(2v+3w)$ in terms of v and w.

$$A(2v+3w) = A(2v) + A(3w) = 2Av + 3Aw = 8v - 3w.$$

b) Suppose that for two 5×5 matrices A and B, we have

$$\det(A) = 3, \quad \det(A^{-1}B) = 7.$$

Find $\det(B)$.

$$\det(B) = \det(AA^{-1}B) = \det(A)\det(A^{-1}B) = 3 \cdot 7 = 21.$$

c) Suppose that A is a positive stochastic matrix with steady-state vector $\begin{pmatrix} 3/10 \\ 7/10 \end{pmatrix}$. What vector does $A^n \begin{pmatrix} 350 \\ 50 \end{pmatrix}$ approach as n becomes very large? Fully simplify your answer.

The sum of entries of $\begin{pmatrix} 350 \\ 50 \end{pmatrix}$ is 400, so our answer is

$$400 \begin{pmatrix} 3/10 \\ 7/10 \end{pmatrix} = \begin{pmatrix} 400 \cdot 3/10 \\ 400 \cdot 7/10 \end{pmatrix} = \begin{pmatrix} 120 \\ 280 \end{pmatrix}.$$

d) Compute the orthogonal projection of $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ onto $\operatorname{Span}\left\{\begin{pmatrix} 2 \\ -3 \end{pmatrix}\right\}$.

For $u = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$, the matrix for projection onto $\operatorname{Span}\{u\}$ is

$$\frac{1}{u \cdot u} uu^T = \frac{1}{4+9} \begin{pmatrix} 2 \\ -3 \end{pmatrix} \begin{pmatrix} 2 & -3 \end{pmatrix} = \frac{1}{13} \begin{pmatrix} 4 & -6 \\ -6 & 9 \end{pmatrix},$$

so

$$\operatorname{proj}_{\operatorname{Span}\{u\}} \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \frac{1}{13} \begin{pmatrix} 4 & -6 \\ -6 & 9 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \frac{1}{13} \begin{pmatrix} -16 \\ 24 \end{pmatrix} = \begin{pmatrix} -16/13 \\ 24/13 \end{pmatrix}.$$

Problem 8.

Consider the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 4 & 4 \\ 1 & 1 & 1 \end{pmatrix}$$

a) Find the characteristic polynomial and the eigenvalues of A.

b) For each eigenvalue of A, find one corresponding eigenvector.

c) Find an invertible 3×3 matrix C and a diagonal matrix D so that $A = CDC^{-1}$.

Solution.

a) The characteristic polynomial is

$$\det(A - \lambda I) = \det \begin{pmatrix} 1-\lambda & 0 & 0 \\ 2 & 4-\lambda & 4 \\ 1 & 1 & 1-\lambda \end{pmatrix} = (1-\lambda)\Big((4-\lambda)(1-\lambda) - 4\Big) = (1-\lambda)\Big(\lambda^2 - 5\lambda\Big)$$

$$= (1-\lambda)(\lambda)(\lambda - 5)$$

so the eigenvalues of A are $\lambda = 0$, $\lambda = 1$, and $\lambda = 5$.

b)

$$\Big(A \mid 0\Big) = \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 2 & 4 & 4 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right) \xrightarrow{RREF} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \text{ so a 0-eigenvector is } \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}.$$

$$\Big(A - I \mid 0\Big) = \left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 2 & 3 & 4 & 0 \\ 1 & 1 & 0 & 0 \end{array}\right) \xrightarrow{RREF} \left(\begin{array}{ccc|c} 1 & 0 & -4 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \text{ so a 1-eigenvector is } \begin{pmatrix} 4 \\ -4 \\ 1 \end{pmatrix}.$$

$$\Big(A - 5I \mid 0\Big) = \left(\begin{array}{ccc|c} -4 & 0 & 0 & 0 \\ 2 & -1 & 4 & 0 \\ 1 & 1 & -4 & 0 \end{array}\right) \xrightarrow{RREF} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \text{ so a 5-eigenvector is } \begin{pmatrix} 0 \\ 4 \\ 1 \end{pmatrix}.$$

c) $A = CDC^{-1}$ where $C = \begin{pmatrix} 0 & 4 & 0 \\ -1 & -4 & 4 \\ 1 & 1 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{pmatrix}$. Of course, there are other possibilities, since the order of eigenvectors in C doesn't matter as long as it is matched by the corresponding eigenvalues in D.

Problem 9.

Let $W = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \text{ in } \mathbf{R}^3 \;\middle|\; x_1 + x_2 + x_3 = 0 \right\}$ and $x = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$.

(i) Find a basis for W.

$W = \mathrm{Nul}\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$, so $x_1 = -x_2 - x_3$ while x_2 and x_3 are free.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}. \qquad \text{Basis} : \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

(ii) Find x_W, the orthogonal projection of x onto W.

We solve $A^T A v = A^T x$ where $A = \begin{pmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$.

$$A^T A = \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \qquad A^T x = \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}.$$

$$\left(A^T A \mid A^T x \right) = \left(\begin{array}{cc|c} 2 & 1 & -1 \\ 1 & 2 & -1 \end{array} \right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{cc|c} 1 & 2 & -1 \\ 2 & 1 & -1 \end{array} \right) \xrightarrow{R_2 = R_2 - 2R_1} \left(\begin{array}{cc|c} 1 & 2 & -1 \\ 0 & -3 & 1 \end{array} \right) \xrightarrow[\text{then } R_2 = -R_2/3]{R_1 = R_1 + 2R_2/3} \left(\begin{array}{cc|c} 1 & 0 & -1/3 \\ 0 & 1 & -1/3 \end{array} \right), \quad v = \begin{pmatrix} -1/3 \\ -1/3 \end{pmatrix}.$$

Thus

$$x_W = Av = \begin{pmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1/3 \\ -1/3 \end{pmatrix} = \begin{pmatrix} 2/3 \\ -1/3 \\ -1/3 \end{pmatrix}.$$

(iii) Find $x_{W^\perp}$

$$x_{W^\perp} = x - x_W = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 2/3 \\ -1/3 \\ -1/3 \end{pmatrix} = \begin{pmatrix} 4/3 \\ 4/3 \\ 4/3 \end{pmatrix}.$$

Problem 10.

Use least squares to find the best-fit line $y = Mx + B$ for the data points

$$(0,0), \qquad (1,2), \qquad (3,-1).$$

Enter your answer below:

$$y = \underline{\hspace{3em}}x \; + \; \underline{\hspace{3em}}.$$

You must show appropriate work. If you simply guess a line or estimate the equation for the line based on the data points, you will receive little or no credit, even if your answer is correct or nearly correct.

No line goes through all three points. The corresponding (inconsistent) system is

$$\begin{aligned} 0 &= M(0)+B \\ 2 &= M(1)+B \\ -1 &= M(3)+B \end{aligned}$$

and the corresponding matrix equation is $Ax = b$ where $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 3 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$.

We solve $A^T A\widehat{x} = A^T b$.

$$A^T A = \begin{pmatrix} 0 & 1 & 3 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 10 & 4 \\ 4 & 3 \end{pmatrix}, \qquad A^T b = \begin{pmatrix} 0 & 1 & 3 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

$$\left(A^T A \,\middle|\, A^T b \right) = \left(\begin{array}{cc|c} 10 & 4 & -1 \\ 4 & 3 & 1 \end{array}\right) \xrightarrow{R_1 = R_1/10} \left(\begin{array}{cc|c} 1 & 4/10 & -1/10 \\ 4 & 3 & 1 \end{array}\right) \xrightarrow{R_2 = R_2 - 4R_1} \left(\begin{array}{cc|c} 1 & 4/10 & -1/10 \\ 0 & 14/10 & 14/10 \end{array}\right)$$

$$\xrightarrow{R_2 = 10R_2/14} \left(\begin{array}{cc|c} 1 & 4/10 & -1/10 \\ 0 & 1 & 1 \end{array}\right) \xrightarrow{R_1 = R_1 - 4R_2/10} \left(\begin{array}{cc|c} 1 & 0 & -5/10 \\ 0 & 1 & 1 \end{array}\right).$$

Thus $\widehat{x} = \begin{pmatrix} -1/2 \\ 1 \end{pmatrix}$. The line is

$$y = \frac{-x}{2} + 1.$$

Scrap paper. This page will not be graded under any circumstances