MATH 1553, FINAL EXAM
SPRING 2024

Name		GT ID	

Circle your instructor and lecture below. Some professors teach more than one lecture, so be sure to circle the correct choice!

Jankowski (A and HP, 8:25-9:15 AM) Jankowski (G, 12:30-1:20 PM)

Hausmann (I, 2:00-2:50 PM) Sanchez-Vargas (M, 3:30-4:20 PM)

Athanasouli (N and PNA, 5:00-5:50 PM) OR: *Advanced Standing Student*

Please **read all instructions** carefully before beginning.

- Write your initials at the top of each page.
- The maximum score on this exam is 100 points, and you have 170 minutes to complete this exam. Each problem is worth 10 points.
- Unless stated otherwise, **the entries of all matrices on the exam are real numbers**.
- There are no calculators or aids of any kind (notes, text, etc.) allowed.
- As always, RREF means “reduced row echelon form.”
- The “zero vector” in $\mathbf{R}^n$ is the vector in $\mathbf{R}^n$ whose entries are all zero.
- On free response problems, show your work, unless instructed otherwise. A correct answer without appropriate work may receive little or no credit!
- We will hand out loose scrap paper, but it **will not be graded** under any circumstances. All answers and work must be written on the exam itself, with no exceptions.
- This exam is double-sided. You should have more than enough space to do every problem on the exam, but if you need extra space, you may use the *back side of the very last page of the exam*. If you do this, you must clearly indicate it.
- You may cite any theorem proved in class or in the sections we covered in the text.

Please read and sign the following statement.

I, the undersigned, hereby affirm that I will not share the contents of this exam with anyone. Furthermore, I have not received inappropriate assistance in the midst of nor prior to taking this exam. I will not discuss this exam with anyone in any form until after 8:50 PM on Tuesday, April 30.

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Problem 1.

[1 point each]

TRUE or FALSE. Circle **T** if the statement is *always* true. Otherwise, answer **F**. You do not need to show work or justify your answer. As stated in the instructions, **the entries of all matrices on the exam are real numbers** unless stated otherwise.

a) **T** **F** Suppose $\{v_1, v_2, v_3, v_4, v_5\}$ is a basis for $\mathbf{R}^n$. Then $n = 5$.

b) **T** **F** The set $W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ in } \mathbf{R}^3 \ : \ x - y + z = 3 \right\}$ is a subspace of $\mathbf{R}^3$.

c) **T** **F** Suppose $T : \mathbf{R}^{20} \to \mathbf{R}^7$ is a linear transformation with standard matrix A, so $T(x) = Ax$. Then $\dim(\text{Nul } A) \geq 13$.

d) **T** **F** Let A be an $m \times n$ matrix, and let T be the corresponding matrix transformation $T(x) = Ax$. If $m > n$, then T cannot be onto.

e) **T** **F** There is a 3×3 matrix A, whose entries are real numbers, so that $2 - i$ and $3i$ are eigenvalues of A.

f) **T** **F** Every nonzero vector in $\mathbf{R}^3$ is an eigenvector of the 3×3 identity matrix.

g) **T** **F** Suppose that u and v are vectors in the 4-eigenspace of some $n \times n$ matrix A. Then $4u - 3v$ must also be in the 4-eigenspace of A.

h) **T** **F** Let A be a 3×3 matrix with characteristic polynomial

$$\det(A - \lambda I) = (1 - \lambda)(3 - \lambda)^2,$$

and suppose $A \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$ and $A \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix}$.

Then A must be diagonalizable.

i) **T** **F** Suppose that W is a subspace of $\mathbf{R}^n$ and that u is a vector in W. Then the orthogonal projection of u onto W is the zero vector.

j) **T** **F** Suppose A is an $m \times n$ matrix and b is a vector in the column space of A. Then every solution to $Ax = b$ is also a least squares solution to $Ax = b$.

Problem 2.

Short answer. You do not need to show your work on this page, and (a)-(d) are unrelated.

a) (3 points) Let $V = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \text{ in } \mathbf{R}^2 \;\middle|\; x^2 + y^2 \leq 5 \right\}$. Answer the following questions.

(i) Does V contain the zero vector? YES NO

(ii) Is V closed under addition? In other words, if u and v are in V, must it be true that $u + v$ is in V? YES NO

(iii) Is V closed under scalar multiplication? In other words, if c is a real number and u is in V, must it be true that cu is in V? YES NO

b) (3 points) Suppose that A is a 2024×100 matrix.
Which of the following are **possible**? Clearly circle all that apply.

(i) The dimension of Row(A) is 105.

(ii) The dimension of Nul(A) is 105.

(iii) The transformation $T(x) = Ax$ is one-to-one.

c) (2 points) Let A be a 35×50 matrix that has 30 pivots. Which one of the following describes the null space of A? Clearly circle your answer.

(i) Nul(A) is a 20-dimensional subspace of $\mathbf{R}^{35}$.

(ii) Nul(A) is a 20-dimensional subspace of $\mathbf{R}^{50}$.

(iii) Nul(A) is a 5-dimensional subspace of $\mathbf{R}^{35}$.

(iv) Nul(A) is a 30-dimensional subspace of $\mathbf{R}^{50}$.

d) (2 points) Let $W = \text{Span}\left\{ \begin{pmatrix} 1 \\ -1 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix} \right\}$, and let $v = \begin{pmatrix} 2 \\ 1 \\ 1 \\ -1 \end{pmatrix}$.

Which of the following statements are true? Clearly circle all that apply.

(i) v is in $W^{\perp}$.

(ii) The set $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$ is a basis for W.

Problem 3.

Short answer. Parts (a)-(c) are unrelated. You do not need to show your work on this page, and there is no partial credit except on part (c).

a) (4 points) Suppose $T : \mathbf{R}^a \to \mathbf{R}^b$ is a linear transformation. Which of the following conditions guarantee that T is onto? Clearly circle **all** that apply.

(i) For each x in $\mathbf{R}^a$, there is at most one y in $\mathbf{R}^b$ so that $T(x) = y$.

(ii) For each y in $\mathbf{R}^b$, there is at least one x in $\mathbf{R}^a$ so that $T(x) = y$.

(iii) For each y in $\mathbf{R}^b$, there is exactly one x in $\mathbf{R}^a$ so that $T(x) = y$.

(iv) For each x in $\mathbf{R}^a$, there is exactly one y in $\mathbf{R}^b$ so that $T(x) = y$.

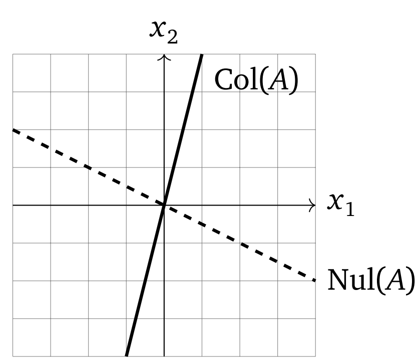
b) (3 points) Which of the following matrices A are invertible? Clearly circle **all** that apply.

(i) The 2×2 matrix A that rotates vectors in $\mathbf{R}^2$ by 30 degrees counterclockwise.

(ii) Any 3×3 matrix A that has eigenvalues $\lambda = 1$, $\lambda = -1$, and $\lambda = 3$.

(iii) $A = \begin{pmatrix} 1 & -1 & 1 \\ 2 & -2 & 2 \\ 0 & 1 & 1 \end{pmatrix}$

c) (3 points) Write a matrix A so that Col(A) is the **solid** line below and Nul(A) is the **dashed** line below.



Enter your answer here: $A = \begin{pmatrix} \quad \end{pmatrix}$

Problem 4.

Short answer. Parts (a) through (d) are unrelated. You do not need to show your work on this page, and there is no partial credit.

a) (2 points) Suppose $\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = 1$. Find $\det\begin{pmatrix} d & e & f \\ 3d-4a & 3e-4b & 3f-4c \\ g & h & i \end{pmatrix}$.

Clearly circle your answer below.

(i) 1 (ii) -1 (iii) 3 (iv) -3 (v) 4 (vi) -4

(vii) 12 (viii) -12 (ix) -24 (x) none of these

b) (2 points) Find the area of the triangle with vertices $(-1,4)$, $(2,-2)$, and $(4,-1)$.

(i) $3/2$ (ii) 3 (iii) $15/2$ (iv) 15 (v) 30 (vi) $45/2$

(vii) 45 (viii) 50 (ix) 60 (x) none of these

c) (2 points) Suppose A and B are 2×2 matrices satisfying $\det(A) = 4$ and $\det(B) = 2$. Find $\det(-3A^{-1}B)$.

(i) $-3/2$ (ii) $9/2$ (iii) $-27/2$ (iv) -24 (v) 24

(vi) -72 (vii) $-9/2$ (viii) 72 (ix) none of these

d) (4 points) Suppose A is a 4×4 matrix with characteristic polynomial

$$\det(A-\lambda I) = (2-\lambda)^2(3-\lambda)(-1-\lambda).$$

Which of the following statements are true? Clearly circle all that apply.

(i) A is invertible.

(ii) If the 2-eigenspace of A is a plane, then A must be diagonalizable.

(iii) It is possible that the 3-eigenspace of A is a plane.

(iv) The zero vector is **not** an eigenvector of A.

Problem 5.

Short answer. You do not need to show your work on this page, and there is no partial credit. Parts (a)-(d) are unrelated.

a) (3 points) Let A be a 2×2 matrix whose entries are real numbers. Which of the following statements must be true? Clearly circle all that apply.

(i) If A has $\lambda = -5$ as an eigenvalue with algebraic multiplicity 2, then -5 is the only eigenvalue of A.

(ii) If $\lambda = 1 - 4i$ is an eigenvalue of A, then A does not have any real eigenvalues.

(iii) If A is a stochastic matrix, then the only eigenvalue of A is $\lambda = 1$.

b) (3 points) Let A be the 2×2 matrix that reflects vectors $\begin{pmatrix} x \\ y \end{pmatrix}$ across the line $y = 3x$. Which of the following are true? Clearly circle all that apply.

(i) The eigenvalues of A are $\lambda = 0$ and $\lambda = 1$.

(ii) A is diagonalizable.

(iii) $A\begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$.

c) (2 points) Let $A = \begin{pmatrix} 0.3 & 0.1 \\ 0.7 & 0.9 \end{pmatrix}$. It has the property that $A\begin{pmatrix} 1 \\ 7 \end{pmatrix} = \begin{pmatrix} 1 \\ 7 \end{pmatrix}$.

What vector does $A^n \begin{pmatrix} 400 \\ 0 \end{pmatrix}$ approach as n gets large? Clearly circle your answer.

(i) $\begin{pmatrix} 64 \\ 336 \end{pmatrix}$ (ii) $\begin{pmatrix} 1/8 \\ 7/8 \end{pmatrix}$ (iii) $\begin{pmatrix} 280 \\ 120 \end{pmatrix}$ (iv) $\begin{pmatrix} 50 \\ 350 \end{pmatrix}$

(v) $\begin{pmatrix} 120 \\ 280 \end{pmatrix}$ (vi) $\begin{pmatrix} 280 \\ 120 \end{pmatrix}$ (vii) $\begin{pmatrix} 50 \\ 0 \end{pmatrix}$ (viii) $\begin{pmatrix} 400 \\ 0 \end{pmatrix}$

d) (2 points) Suppose W is a subspace of $\mathbf{R}^n$ and B is the matrix for orthogonal projection onto W. Which of the following must be true? Clearly circle all that apply.

(i) The eigenvalues of B are $\lambda = -1$ and $\lambda = 1$.

(ii) $B^3 = B$.

Problem 6.

Short answer. You do not need to show your work on this page, and (a)-(c) are unrelated.

a) (2 points) Find the value of c so that the vectors $\begin{pmatrix} 1 \\ -2 \\ c \\ 4 \end{pmatrix}$ and $\begin{pmatrix} 6 \\ c \\ 0 \\ 2 \end{pmatrix}$ are orthogonal.

Fill in the blank: $c =$ ______________.

b) (5 points) Suppose W is a subspace of $\mathbf{R}^3$ and x is a vector so that

$$x_W = \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix} \quad \text{and} \quad x_{W^\perp} = \begin{pmatrix} 5 \\ 2 \\ -1 \end{pmatrix}.$$

(i) What is the distance from x to W? Clearly circle your answer.

$\sqrt{11}$ $\qquad \sqrt{3}$ $\qquad \sqrt{30}$ $\qquad \sqrt{6}$ $\qquad \sqrt{41}$ $\qquad 11$ $\qquad 3$ $\qquad 30$ $\qquad 41$

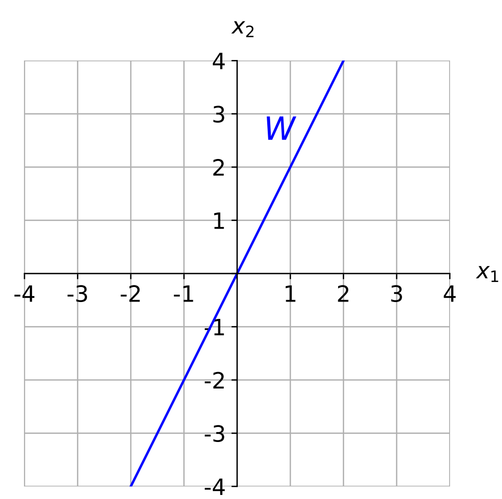
(ii) What is the closest vector to x in W? Clearly circle your answer.

$$\begin{pmatrix} 1 \\ -3 \\ -1 \end{pmatrix} \qquad \begin{pmatrix} -6 \\ 1 \\ 2 \end{pmatrix} \qquad \begin{pmatrix} 5 \\ 2 \\ -1 \end{pmatrix} \qquad \begin{pmatrix} 4 \\ 5 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}$$

(iii) Which **one** of the following **could** be $W^\perp$? Clearly circle your answer.

$$\mathrm{Nul}\begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix} \qquad \mathrm{Col}\begin{pmatrix} 5 \\ 2 \\ -1 \end{pmatrix} \qquad \mathrm{Row}\begin{pmatrix} 5 \\ 2 \\ -1 \end{pmatrix} \qquad \mathrm{Nul}\begin{pmatrix} 5 & 2 & -1 \end{pmatrix} \qquad \mathrm{Row}\begin{pmatrix} -1 & 3 & 1 \end{pmatrix}$$

c) (3 points) Let W be the line graphed below, and let $x = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$. On the graph below, very carefully **draw** and **label** x, x_W, and $x_{W^\perp}$.



Problem 7.

Free response. Show your work unless otherwise indicated! A correct answer without sufficient work may receive little or no credit.

For this problem, let $A = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 6 & -24 & 18 \end{pmatrix}$.

a) (2 pts) Write the eigenvalues of A. You do not need to show your work on this part.

Fill in the blank: the eigenvalues are ____________________.

b) (6 points) For each eigenvalue of A, find a basis for the corresponding eigenspace.

c) (2 points) The matrix A is diagonalizable. Write a 3×3 matrix C and a 3×3 diagonal matrix D so that $A = CDC^{-1}$. Enter your answer below.

$$C = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix} \qquad D = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$$

Problem 8.

Free response. Show your work! A correct answer without sufficient work may receive little or no credit.

a) Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the transformation corresponding to reflection over the line $y = x$. Find the standard matrix A for T, so $T(v) = Av$. Enter your answer below.

$$A = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$$

b) Let $S : \mathbf{R}^3 \to \mathbf{R}^2$ be the transformation defined by $S(x, y, z) = (3x + y, -y + 2z)$. Find the standard matrix B for S, so $S(v) = Bv$. Enter your answer below.

$$B = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$$

c) Which of the following expressions are possible to calculate? Clearly circle all that apply. You do not need to show your work on this part.

$$T\begin{pmatrix} 3 \\ -5 \\ 8 \end{pmatrix} \qquad S\begin{pmatrix} 6 \\ 2 \\ -1 \end{pmatrix} \qquad (T \circ S)\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \qquad (S \circ T)\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

d) Which one of the following compositions makes sense? Circle the **one** correct answer. You do not need to show your work on this part.

$$T \circ S \qquad\qquad S \circ T$$

e) Find the standard matrix C for the transformation you circled in part (d). Enter your answer below.

$$C = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$$

Problem 9.

Free response. Show your work! A correct answer without sufficient work may receive little or no credit. Parts (a) and (b) are unrelated.

a) Let $W = \text{Span}\left\{\begin{pmatrix}-1\\2\\5\end{pmatrix}\right\}$. Find a basis for $W^{\perp}$.

b) Let $W = \text{Span}\left\{\begin{pmatrix}-3\\2\end{pmatrix}\right\}$ and let $x = \begin{pmatrix}2\\1\end{pmatrix}$.

Find x_W (the orthogonal projection of x onto W) and $x_{W^{\perp}}$. Enter your answers below. **Simplify all fractions in your answer as much as possible**.

$$x_W = \begin{pmatrix} \\ \\ \end{pmatrix} \qquad\qquad x_{W^{\perp}} = \begin{pmatrix} \\ \\ \end{pmatrix}.$$

Problem 10.

Free response. Show your work!
Use least squares to find the best-fit line $y = Mx + B$ for the data points

$$(0, 3), \qquad (2, -7), \qquad (4, -5).$$

Enter your answer below:

$$y = \underline{\qquad\quad} x \ + \ \underline{\qquad\quad}.$$

You **must** show appropriate work using least squares. If you simply guess a line or estimate the equation for the line based on the data points, you will receive little or no credit, even if your answer is correct or nearly correct.

This page is reserved ONLY for work that did not fit elsewhere on the exam.

If you use this page, please clearly indicate (on the problem's page and here) which problems you are doing.