MATH 1553, SPRING 2023
FINAL EXAM

Name		GT ID	

Circle your lecture below.

Jankowski, lec. A and HP (8:25-9:15 AM) Jankowski, lecture D (9:30-10:20 AM)

Sane, lecture G (12:30-1:20 PM)

Sun, lecture I (2:00-2:50 PM) Sun, lecture M (3:30-4:20 PM)

Please **read all instructions** carefully before beginning.

- Write your initials at the top of each page.
- The maximum score on this exam is 100 points, and you have 170 minutes to complete this exam. Each problem is worth 10 points.
- There are no calculators or aids of any kind (notes, text, etc.) allowed.
- Simplify your answers as much as possible. For example, you may lose points if you do not simplify $\frac{8}{2}$ to 4, or if you do not simplify $\frac{0.1}{0.9}$ to $\frac{1}{9}$, etc.
- As always, RREF means "reduced row echelon form." The "zero vector" in $\mathbf{R}^n$ is the vector in $\mathbf{R}^n$ whose entries are all zero.
- Show your work, unless instructed otherwise. A correct answer without appropriate work will receive little or no credit!
- Unless stated otherwise, **the entries of all matrices on the exam are real numbers**.
- We use $e_1, e_2, \ldots, e_n$ to denote the standard unit coordinate vectors of $\mathbf{R}^n$.
- We will hand out loose scrap paper, but it **will not be graded**. All answers and all work must be written on the exam itself, with no exceptions.
- This exam is double-sided. You should have more than enough space to do every problem on the exam, but if you need extra space, you may use the *back side of the very last page of the exam*. If you do this, you must clearly indicate it.
- You may cite any theorem proved in class or in the sections we covered in the text.

Please read and sign the following statement.

I, the undersigned, hereby affirm that I will not share the contents of this exam with anyone. Furthermore, I have not received inappropriate assistance in the midst of nor prior to taking this exam. I will not discuss this exam with anyone in any form until after 9:00 PM on Tuesday, May 2.

____

Problem 1.

True or false. If the statement is *ever* false, circle FALSE. You do not need to show any work, and there is no partial credit. Each question is worth 1 point.

a) Suppose W is a subspace of $\mathbf{R}^7$ and that there is a basis of W consisting of 4 vectors. Then $\dim(W) = 4$.

TRUE FALSE

b) If A is a 5×3 matrix, then the equation $Ax = 0$ must have infinitely many solutions.

TRUE FALSE

c) The set $W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ in } \mathbf{R}^3 \;\middle|\; x - y - z = 3 \right\}$ is a subspace of $\mathbf{R}^3$.

TRUE FALSE

d) Suppose A is an $n \times n$ matrix and $\det(A) = 0$. Then the RREF of A has a row whose entries are all 0.

TRUE FALSE

e) Let A be a 3×3 diagonalizable matrix. If the only eigenvalue of A is $\lambda = 1$, then A must be the 3×3 identity matrix.

TRUE FALSE

f) Suppose that $T : \mathbf{R}^3 \to \mathbf{R}^3$ is a matrix transformation $T(x) = Ax$, and suppose that the range of T is a line. Then $\lambda = 0$ must be an eigenvalue of A.

TRUE FALSE

g) If v and w are eigenvectors of a 3×3 matrix A, then $v + w$ must be an eigenvector of A.

TRUE FALSE

h) Suppose u, v, and w are vectors in $\mathbf{R}^n$. If u is orthogonal to v and u is orthogonal to w, then u must also be orthogonal to $v - w$.

TRUE FALSE

i) Suppose A is an $m \times n$ matrix and x is a vector in $\mathbf{R}^m$. If x is in the column space of A, then $x \cdot v = 0$ for every vector v in the null space of A^T.

TRUE FALSE

j) There is a 2×2 positive stochastic matrix A so that $\lambda = i/2$ is an eigenvalue of A.

TRUE FALSE

Problem 2.

Multiple choice and short answer. Parts (a) through (d) are unrelated. You do not need to show your work on this page, and there is no partial credit.

a) (3 points) Let $V = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \text{ in } \mathbf{R}^2 \;\middle|\; x - y \geq 0 \right\}$. Determine which properties of a subspace of $\mathbf{R}^2$ are satsified by W to answer the questions below.

(i) Does V contain the zero vector? YES NO

(ii) Is V closed under addition? In other words, if u and v are vectors in V, must it be true that $u + v$ is also in V? YES NO

(iii) Is V closed under scalar multiplication? In other words, if c is a real number and u is in V, must it be true that cu is also in V? YES NO

b) Let $A = \begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}$.

(i) (2 points) What is A^{-1}? Clearly circle your answer below.

$$A^{-1} = \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix} \qquad A^{-1} = \begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix} \qquad A^{-1} = \begin{pmatrix} 3 & -1 \\ 1 & 2 \end{pmatrix}$$

$$A^{-1} = \frac{1}{7}\begin{pmatrix} 3 & -1 \\ 1 & 2 \end{pmatrix} \qquad A^{-1} = \frac{1}{7}\begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix} \qquad A^{-1} = \frac{1}{7}\begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}$$

(ii) (1 point) Suppose S is a rectangle in $\mathbf{R}^2$ with area 3, and let T be the matrix transformation $T(x) = Ax$, where A is the matrix in part (i) above.
Fill in the blank: the area of $T(S)$ is ________.

c) (2 points) Let $B = \begin{pmatrix} 1 & c & 0 \\ 0 & 2 & c \\ 0 & 0 & 1 \end{pmatrix}$, where c is a real number. Which of the following statements must be true? Clearly select all that apply.

(i) $\mathrm{Col}(B) = \mathbf{R}^3$, no matter what the value of c is.

(ii) If $c = 1$, then B is diagonalizable.

d) (2 points) Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & -1 \end{pmatrix}$. Which one of the following is a basis for $(\mathrm{Col}\ A)^{\perp}$?

(I) $\left\{ \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \right\}$ (II) $\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right\}$ (III) $\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} \right\}$ (IV) $\left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$ (V) none of these

Problem 3.

Multiple choice and short answer. Parts (a), (b), and (c) are unrelated. You do not need to show your work on this problem, and there is no partial credit.

a) (2 points) Suppose A is a 25×20 matrix and the dimension of the column space of A is 6. Which *one* of the following describes the null space of A?

(i) Nul(A) is a 19-dimensional subspace of $\mathbf{R}^{25}$.

(ii) Nul(A) is a 14-dimensional subspace of $\mathbf{R}^{25}$.

(iii) Nul(A) is a 19-dimensional subspace of $\mathbf{R}^{20}$.

(iv) Nul(A) is a 14-dimensional subspace of $\mathbf{R}^{20}$.

b) (4 points) Suppose A is a 4×4 matrix with characteristic polynomial

$$\det(A - \lambda I) = (\lambda - 1)^2(\lambda + 2)^2.$$

Which of the following statements are true? Clearly circle all that apply.

(i) The columns of A are linearly independent.

(ii) $\det(A) = 4$.

(iii) If the 1-eigenspace of A is a line, then A is not diagonalizable.

(iv) The matrix equation $Ax = -2x$ has infinitely many solutions.

c) (4 points) Let A be an 5×4 matrix and let T be the matrix transformation $T(x) = Ax$. Which of the following conditions *guarantee* that T is one-to-one? Clearly circle all that apply.

(i) For each x in $\mathbf{R}^4$, there is at most one vector y in $\mathbf{R}^5$ so that $T(x) = y$.

(ii) For each y in $\mathbf{R}^5$, there is at most one vector x in $\mathbf{R}^4$ so that $T(x) = y$.

(iii) The set $\{Ae_1, Ae_2, Ae_3, Ae_4\}$ is linearly independent.

(iv) The homogeneous matrix equation $Ax = 0$ has only the trivial solution.

Problem 4.

Short answer and multiple choice. Parts (a), (b), (c), and (d) are unrelated. You do not need to show your work on this problem.

a) (3 points) Match each matrix below with its corresponding transformation (choosing from (i) through (viii)) by clearly writing that roman numeral next to the matrix. Note there are three matrices and eight options, so not every option is used.

$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is the standard matrix for ________

$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is the standard matrix for ________

$\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$ is the standard matrix for ________

(i) Reflection across the x-axis in $\mathbf{R}^2$.
(ii) Reflection across the y-axis in $\mathbf{R}^2$.
(iii) Reflection across the line $y = x$ in $\mathbf{R}^2$.
(iv) Reflection across the line $y = -x$ in $\mathbf{R}^2$.
(v) Rotation counterclockwise by $\pi/4$ radians in $\mathbf{R}^2$.
(vi) Rotation clockwise by $\pi/4$ radians in $\mathbf{R}^2$.
(vii) Rotation counterclockwise by $\pi/2$ radians in $\mathbf{R}^2$.
(viii) Rotation clockwise by $\pi/2$ radians in $\mathbf{R}^2$.

b) (3 points) Which of the following linear transformations are onto? Circle all that apply.

(i) $T : \mathbf{R}^3 \to \mathbf{R}^2$ defined by $T(x, y) = (x + y, x + y)$.

(ii) $T : \mathbf{R}^2 \to \mathbf{R}^3$ defined by $T(x, y) = (x, y, x - y)$.

(iii) The transformation $T : \mathbf{R}^2 \to \mathbf{R}^2$ that reflects vectors across the x-axis.

c) (2 points) Let $A = \begin{pmatrix} 1 & 5 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1/4 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 5 \\ 2 & 1 \end{pmatrix}^{-1}$. Find $A\begin{pmatrix} 5 \\ 1 \end{pmatrix}$.

Enter your answer in the space to the right: $A\begin{pmatrix} 5 \\ 1 \end{pmatrix} = \begin{pmatrix} \quad \\ \quad \end{pmatrix}$

d) (2 points) Suppose A is the matrix for the orthogonal projection onto Span$\{v\}$, where v is a nonzero vector in $\mathbf{R}^4$. Which of the following are true? Clearly circle all that apply.

(i) $Av = v$.

(ii) The null space of A is 3-dimensional.

Problem 5.

Short answer and multiple choice. Parts (a), (b), (c), and (d) are unrelated. You do not need to show your work on this problem, and there is no partial credit.

a) Let $S : \mathbf{R}^3 \to \mathbf{R}^2$ be the transformation given by

$$S\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x - 2y \\ x - y - z \end{pmatrix},$$

and let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the transformation that reflects vectors across the line $y = x$.

(i) (1 point) Let A be the standard matrix for S. What is the dimension of the null space of A? Clearly circle your answer below.

0 1 2 3

(ii) (1 point) Let B be the standard matrix for the composition $T \circ S$. How many columns does B have? Clearly circle your answer below.

1 2 3 4

b) (2 points) Find the value of c so that $\begin{pmatrix} 8 \\ -2 \\ c \end{pmatrix}$ and $\begin{pmatrix} 1 \\ c \\ 3 \end{pmatrix}$ are orthogonal.

Enter your answer here: $c =$ ________.

c) (3 points) Suppose A is a 4×4 matrix with 4 **different** real eigenvalues. Which of the following must be true? Clearly circle all that apply.

(i) Each eigenvalue of A has geometric multiplicity 1.

(ii) Every nonzero vector in $\mathbf{R}^4$ is an eigenvector of A.

(iii) A is diagonalizable.

d) (3 points) Suppose A and B are $n \times n$ matrices. Which of the following are true? Clearly circle all that apply.

(i) If $Ax = 0$ has only the trivial solution, then $\det(A) = 0$.

(ii) If A and B are invertible, then so is $A + B$ and $(A + B)^{-1} = A^{-1} + B^{-1}$.

(iii) If $Ax = b$ has infinitely many solutions for some b in $\mathbf{R}^n$, then A is not invertible.

Problem 6.

Parts (a), (b), and (c) are unrelated and are 5 points, 3 points, and 2 points, respectively. You do not need to show your work on this page.

a) Suppose W is a subspace of $\mathbf{R}^3$ and x is a vector in $\mathbf{R}^3$ whose orthogonal decomposition with respect to W is $x = x_W + x_{W^\perp}$ where

$$x_W = \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix} \quad \text{and} \quad x_{W^\perp} = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}.$$

(i) (2 points) What is the closest vector to x in W?

(I) $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ (II) $\frac{1}{\sqrt{17}}\begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix}$ (III) $\frac{1}{3}\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$ (IV) $\begin{pmatrix} 1 \\ -4 \\ 1 \end{pmatrix}$

$\begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix}$ (VI) $\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$ (VII) $\begin{pmatrix} 5 \\ 0 \\ 1 \end{pmatrix}$ (VIII) $\begin{pmatrix} -3 \\ 2 \\ -2 \end{pmatrix}$

(ii) (2 points) What is the length of x?

(I) $\sqrt{17}$ (II) 9 (III) $\sqrt{26}$ (IV) 7

(V) 5 (VI) 17 (VII) 26 (VIII) 3

(iii) (1 pt) Is $\begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix}$ in $W^\perp$? YES NO NOT ENOUGH INFO

b) (3 points) Suppose A is the matrix for the orthogonal projection onto the subspace $W = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \text{ in } \mathbf{R}^3 \;\middle|\; 2x_1 - x_2 - x_3 = 0 \right\}$. Which of the following must be true? Clearly select all that apply.

(i) $A\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

(ii) $A^2 = A$.

(iii) The eigenvalues of A are 0 and 1.

c) (2 pts) Let A be a 2×2 positive stochastic matrix satisfying $A\begin{pmatrix} 1 \\ 9 \end{pmatrix} = \begin{pmatrix} 1 \\ 9 \end{pmatrix}$.

Fill in the blank: As n gets very large, $A^n\begin{pmatrix} 40 \\ 10 \end{pmatrix}$ approaches the vector $\begin{pmatrix} \quad \\ \quad \end{pmatrix}$.

Problem 7.

Free response. You do not need to show your work on parts (a) and (d) of this problem, but show your work on part (b) and justify your answer in part (c).

For this problem, consider the following matrix A and its reduced row echelon form.

$$A = \begin{pmatrix} 1 & -2 & 0 & -1 \\ 2 & -4 & 1 & 1 \\ -2 & 4 & -2 & -4 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & -2 & 0 & -1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

a) (2 points) Write a basis for Col(A).

b) (4 points) Find a basis for Nul(A).

c) (2 points) Write vectors x and y (with $x \neq y$) satisfying $Ax = Ay$. Briefly justify your answer.

d) (2 points) Write a matrix B so that $\mathrm{Col}(B) = \mathrm{Nul}(A)$.

Problem 8.

Free response. Show your work unless instructed otherwise! A correct answer without sufficient work may receive little or no credit. Parts (a) and (b) are unrelated.

a) (5 points) Let $A = \begin{pmatrix} -2 & -4 \\ 8 & 6 \end{pmatrix}$. Find the eigenvalues of A. For the eigenvalue with *negative* imaginary part, find one corresponding eigenvector v. Enter your answers in the space provided below.

The eigenvalues are ______________________. [simplify the eigenvalues completely]

For the eigenvalue with *negative* imaginary part, an eigenvector is $v = \begin{pmatrix} \quad \\ \quad \end{pmatrix}$.

b) (5 points) Let $W = \text{Span}\left\{\begin{pmatrix} 3 \\ -1 \end{pmatrix}\right\}$, and let $x = \begin{pmatrix} 10 \\ 10 \end{pmatrix}$. Find the orthogonal decomposition of x with respect to W. In other words, find x_W in W and $x_{W^\perp}$ in $W^\perp$ so that $x = x_W + x_{W^\perp}$. Write your answers in the space provided below.

$$x_W = \begin{pmatrix} \quad \\ \quad \end{pmatrix} \qquad x_{W^\perp} = \begin{pmatrix} \quad \\ \quad \end{pmatrix}.$$

Problem 9.

Show your work unless instructed otherwise! A correct answer without sufficient work may receive little or no credit.

For this problem, let $A = \begin{pmatrix} 5 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & -3 & 5 \end{pmatrix}$.

a) (2 points) Find all eigenvalues of A and write them in the box below. You do not need to show your work on this part.

b) (5 points) For each of the eigenvalues, find a basis of the corresponding eigenspace.

c) (3 points) A is diagonalizable. Write an invertible 3×3 matrix C and a diagonal matrix D so that $A = CDC^{-1}$. Enter your answer below.

$$C = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix} \qquad D = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}.$$

Problem 10.

Free response. Show your work!

Use least squares to find the best-fit line $y = Mx + B$ for the data points

$$(0, 14), \qquad (2, -5), \qquad (6, -1).$$

Enter your answer below:

$$y = \underline{\qquad\quad} x \ + \ \underline{\qquad\quad}.$$

You must show appropriate work and *simplify your answer completely* (if your answer has fractions, simplify them completely). If you simply guess a line or estimate the equation for the line based on the data points, you will receive little or no credit, even if your answer is correct or nearly correct.

This page is reserved ONLY for work that did not fit elsewhere on the exam.

If you use this page, please clearly indicate (on the problem's page and here) which problems you are doing.