MATH 1553
FINAL EXAM, SPRING 2018

Name	

Circle the name of your instructor below:

Fathi   Jankowski, lecture A   Jankowski, lecture C   Kordek

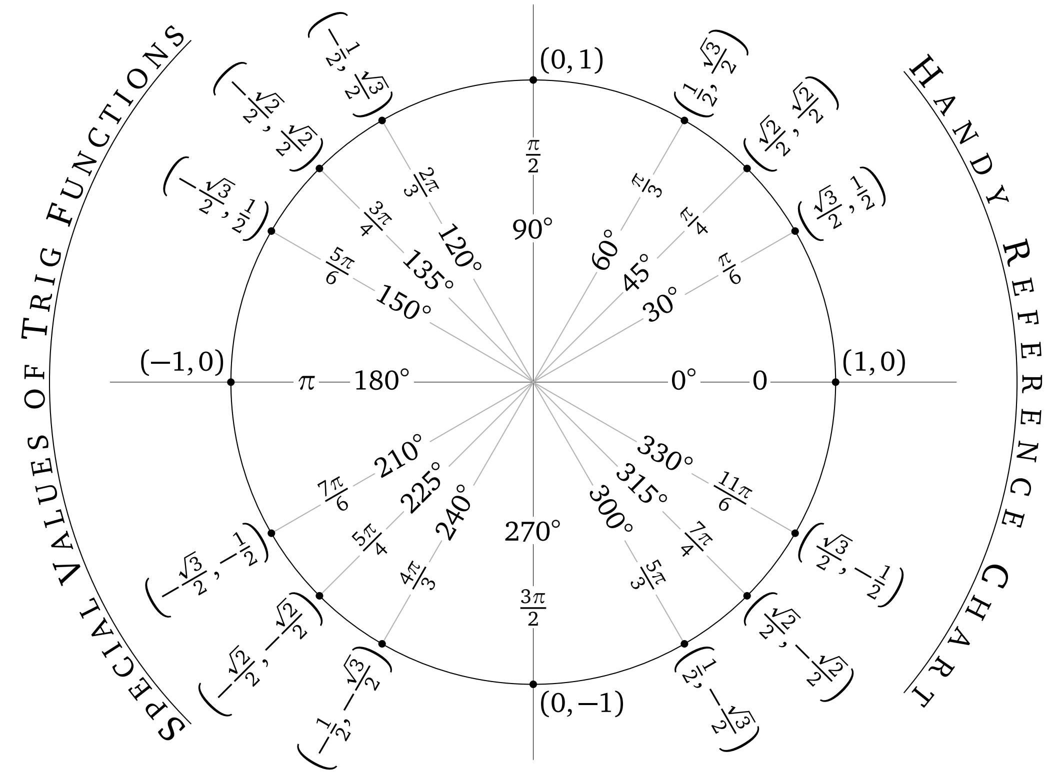
Strenner, lecture H   Strenner, lecture M   Yan

DO NOT WRITE IN THE TABLE BELOW! It will be used to record scores.

1	2	3	4	5	6	7	8	9	10	Total

Please **read all instructions** carefully before beginning.

- The maximum score on this exam is 100 points.
- You have 170 minutes to complete this exam.
- You may not use any calculators or aids of any kind (notes, text, etc.).
- Please show your work. A correct answer without appropriate work will receive little or no credit.
- You may cite any theorem proved in class or in the sections we covered in the text.
- Check your answers if you have time left! Most linear algebra computations can be easily verified for correctness.
- Good luck!



SPECIAL VALUES OF TRIG FUNCTIONS
HANDY REFERENCE CHART
(0,1)
(1,0)
(−1,0)
(0,−1)
0°
0
30°
π/6
45°
π/4
60°
π/3
90°
π/2
120°
2π/3
135°
3π/4
150°
5π/6
180°
π
210°
7π/6
225°
5π/4
240°
4π/3
270°
3π/2
300°
5π/3
315°
7π/4
330°
11π/6
(√3/2, 1/2)
(√2/2, √2/2)
(1/2, √3/2)
(−1/2, √3/2)
(−√2/2, √2/2)
(−√3/2, 1/2)
(−√3/2, −1/2)
(−√2/2, −√2/2)
(−1/2, −√3/2)
(1/2, −√3/2)
(√2/2, −√2/2)
(√3/2, −1/2)

Problem 1.

[2 points each]

True or false. Circle **T** if the statement is *always* true. Otherwise, circle **F**.
You do not need to justify your answer, and there is no partial credit.
In each case, assume that the entries of all matrices and all vectors are real numbers.

a) **T** **F** If A is a 3×4 matrix and b is in $\mathbf{R}^3$, then the set of solutions to $Ax = b$ is a subspace of $\mathbf{R}^4$.

b) **T** **F** If A is a 3×7 matrix then $\text{rank}(A) < \dim(\text{Nul } A)$.

c) **T** **F** Let A be an $n \times n$ matrix. If A has two identical columns, then A is not invertible.

d) **T** **F** If A and B are 2×2 matrices that both have λ as an eigenvalue, then λ^2 is an eigenvalue of AB.

e) **T** **F** If A is an $n \times n$ matrix with n linearly independent eigenvectors, then each eigenvalue of A has algebraic multiplicity 1.

f) **T** **F** The least-squares solution to $Ax = b$ is unique if

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 2 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

g) **T** **F** If v and w are nonzero orthogonal vectors, then $\text{proj}_{\text{Span}\{v\}}\, w$ is the zero vector.

h) **T** **F** If A is a 4×3 matrix and Col A is 2-dimensional, then the orthogonal complement of Col A is also 2-dimensional.

Problem 2. [10 points]

Short answer. On this page, you do not need to show your work. There is no partial credit for (a), (b), or (c).

a) Find $(AB)^{-1}$ if $A^{-1} = \begin{pmatrix} 2 & 0 \\ 3 & 1 \end{pmatrix}$ and $B^{-1} = \begin{pmatrix} -1 & 2 \\ 0 & 5 \end{pmatrix}$.

b) Which of the following are subspaces of $\mathbf{R}^3$? Circle all that apply.

(i) The plane $x - y + z = 1$ in $\mathbf{R}^3$.

(ii) The z-axis in $\mathbf{R}^3$.

(iii) The set of all vectors $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ in $\mathbf{R}^3$ that satisfy $x + 3y = z$.

c) Write a nonzero 2×2 matrix A which is upper-triangular and satisfies $A^2 = 0$.

d) Write three different 3×3 matrices A, B, and C which each have eigenvalue $\lambda = -1$ with algebraic multiplicity 3, so that no two of the different matrices are similar.

Problem 3. [10 points]

Short answer. Show your computations for credit on (b) and (c).

a) Let u and v be orthogonal vectors in $\mathbf{R}^3$ with $||u|| = 5$ and $||v|| = 1$.
Find $u \cdot (u - v)$.

b) Find a nonzero vector v orthogonal to $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix}$.

c) Use row reduction to find the inverse of the matrix

$$M = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

d) In the following questions, b_1 and b_2 are vectors in $\mathbf{R}^3$.
Which statements are possible? Circle all that apply.

(i) b_1 and b_2 are nonzero and orthogonal, but the set $\{b_1, b_2\}$ is linearly dependent.

(ii) $\{b_1, b_2\}$ is a linearly independent set, but b_1 and b_2 are not orthogonal.

Problem 4. [10 points]

Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the transformation that rotates counterclockwise by $\dfrac{\pi}{6}$ radians, and let $U : \mathbf{R}^2 \to \mathbf{R}^2$ be the transformation that reflects about the line $y = x$.

a) Find the standard matrix A for T and the standard matrix B for U.

b) Find the matrix for T^{-1} and the matrix for U^{-1}. Clearly label your answers.

c) Compute the matrix M for the linear transformation from $\mathbf{R}^2$ to $\mathbf{R}^2$ that first rotates *clockwise* by $\frac{\pi}{6}$ radians, then reflects about the line $y = x$, then rotates counterclockwise by $\frac{\pi}{6}$ radians.

Problem 5.

[8 points]

Consider the following matrix A, and its reduced row echelon form.

$$A = \begin{pmatrix} 1 & 0 & 3 & 1 \\ -1 & 4 & -11 & 7 \\ -2 & 3 & -12 & 4 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

a) Find a basis for Col A.

b) Find a basis for Nul A.

c) What is $\dim\left((\mathrm{Nul}A)^{\perp}\right)$? Briefly justify your answer.

Problem 6.

[9 points]

Parts (a) and (b) are unrelated.

a) Compute the determinant of A, where $A = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 1 & 3 \\ 0 & -2 & 1 & 0 \end{pmatrix}$.

b) Let $W = \text{Span}\left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 4 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ -8 \\ -2 \\ 0 \end{pmatrix} \right\}$. Find an orthogonal basis for W.

Problem 7. [10 points]

Consider the matrix $A = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$.

a) Compute the characteristic polynomial of A.

b) Write the eigenvalues of A.

c) For each eigenvalue of A, compute a basis for the corresponding eigenspace.

d) Decide whether A is diagonalizable. If it is diagonalizable, find an invertible 3×3 matrix P and a diagonal matrix D such that $A = PDP^{-1}$. If A is not diagonalizable, explain why.

Problem 8. [10 points]

Let $A = \begin{pmatrix} 2 & -6 \\ 2 & 2 \end{pmatrix}$.

(a) Find the characteristic polynomial of A.

(b) Find the complex eigenvalues of A.

(c) For the eigenvalue with negative imaginary part, find a corresponding eigenvector.

(d) Find a matrix C that represents a composition of scaling and rotation and is similar to A.

(e) What is the scale factor for C?

Problem 9.

[9 points]

Let $W = \mathrm{Span}\{v_1, v_2\}$, where $v_1 = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$.

a) Find the closest point w in W to $x = \begin{pmatrix} 0 \\ 14 \\ -4 \end{pmatrix}$.

b) Find the distance from w to $\begin{pmatrix} 0 \\ 14 \\ -4 \end{pmatrix}$.

c) Find the standard matrix A for the orthogonal projection onto $\mathrm{Span}\{v_1\}$.

Problem 10. [8 points]

Find the best-fit line $y = c + mx$ for the points $(-5, -6)$, $(-2, 9)$, and $(1, 12)$.

Scratch paper. This sheet will not be graded under any circumstances.