MATH 1553, FINAL EXAM
FALL 2023

Name		**GT ID**	

Circle your instructor and lecture below.

Jankowski (A, 8:25-9:15 AM) Kafer (B, 8:25-9:15 AM) Irvine (C, 9:30-10:20)

Kafer (D, 9:30-10:20 AM) He (G, 12:30-1:20 PM) Goldsztein (H, 12:30-1:20)

Goldsztein (I, 2:00-2:50 PM) Neto (L, 3:30-4:20 PM)

Yu (M, 3:30-4:20 PM) Ostrovskii, (N, 5:00-5:50 PM)

Please **read all instructions** carefully before beginning.

- Write your initials at the top of each page. The maximum score on this exam is 100 points, and you have 170 minutes to complete it. Each problem is worth 10 points.
- There are no calculators or aids of any kind (notes, text, etc.) allowed.
- Simplify your answers as much as possible. For example, you may lose points if you do not simplify $\frac{8}{2}$ to 4, or if you do not simplify $\frac{0.1}{0.9}$ to $\frac{1}{9}$, etc.
- As always, RREF means "reduced row echelon form." The "zero vector" in $\mathbf{R}^n$ is the vector in $\mathbf{R}^n$ whose entries are all zero.
- Show your work, unless instructed otherwise. A correct answer without appropriate work will receive little or no credit!
- Unless stated otherwise, **the entries of all matrices on the exam are real numbers**.
- We use $e_1, e_2, \ldots, e_n$ to denote the standard unit coordinate vectors of $\mathbf{R}^n$.
- We will hand out loose scrap paper, but it **will not be graded**. All answers and all work must be written on the exam itself, with no exceptions.
- This exam is double-sided. You should have more than enough space to do every problem on the exam, but if you need extra space, you may use the *back side of the very last page of the exam*. If you do this, you must clearly indicate it.

Please read and sign the following statement.

I, the undersigned, hereby affirm that I will not share the contents of this exam with anyone. Furthermore, I have not received inappropriate assistance in the midst of nor prior to taking this exam. I will not discuss this exam with anyone in any form until after 9:00 PM on Tuesday, December 12.

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Problem 1. [1 point each]

TRUE or FALSE. Circle **T** if the statement is *always* true. Otherwise, answer **F**. You do not need to show work or justify your answer.

a) **T** **F** If A is a 5×3 matrix, then the columns of A must be linearly dependent.

b) **T** **F** Suppose that u and v are vectors in a subspace W of $\mathbf{R}^n$. Then $2u-5v$ must also be a vector in W.

c) **T** **F** If $T:\mathbf{R}^5\to\mathbf{R}^6$ is a linear transformation, then T cannot be onto.

d) **T** **F** If A is a 3×5 matrix and B is a 5×4 matrix, then the linear transformation T given by $T(x)=ABx$ has domain $\mathbf{R}^3$ and codomain $\mathbf{R}^4$.

e) **T** **F** Suppose A is a 4×4 matrix with columns v_1, v_2, v_3, v_4, and suppose

$$v_1-v_2+2v_3-v_4=0.$$

Then A cannot be invertible.

f) **T** **F** Suppose A is a 4×4 matrix with characteristic polynomial

$$\det(A-\lambda I)=(1-\lambda)^3(7-\lambda).$$

Then A must be invertible.

g) **T** **F** If u and v are eigenvectors of a 3×3 matrix A, then $u+v$ must also be an eigenvector of A.

h) **T** **F** Let $A=\begin{pmatrix}4/5 & 1/12\\ 1/5 & 11/12\end{pmatrix}$. Then the 1-eigenspace of A is a line in $\mathbf{R}^2$.

i) **T** **F** Suppose W is a subspace in $\mathbf{R}^n$ and u is a vector in W. Then the orthogonal projection of u onto W is the zero vector.

j) **T** **F** Let $W=\text{Span}\left\{\begin{pmatrix}1\\-2\\0\end{pmatrix}\right\}$, and let B be the matrix for orthogonal projection onto W. Then the null space of B is a line.

Problem 2.

Short answer. You do not need to show your work on this page, and (a)-(d) are unrelated.

a) (2 points) Select the **one** matrix below whose column space and null space satisfy

$$\mathrm{Col}(A) = \mathrm{Span}\left\{\begin{pmatrix}4\\1\end{pmatrix}\right\} \quad \text{and} \quad \mathrm{Nul}(A) = \mathrm{Span}\left\{\begin{pmatrix}-1\\2\end{pmatrix}\right\}.$$

(i) $\begin{pmatrix}4 & -1\\1 & 2\end{pmatrix}$ (ii) $\begin{pmatrix}4 & 8\\1 & 2\end{pmatrix}$ (iii) $\begin{pmatrix}4 & -8\\1 & -2\end{pmatrix}$ (iv) $\begin{pmatrix}4 & 2\\1 & 1/2\end{pmatrix}$

(v) $\begin{pmatrix}4 & -2\\1 & -1/2\end{pmatrix}$ (vi) $\begin{pmatrix}4 & 1\\-1 & 2\end{pmatrix}$ (vii) $\begin{pmatrix}4 & -4\\1 & 2\end{pmatrix}$

b) (3 points) Which of the following are subspaces of $\mathbf{R}^3$? Clearly circle all that apply.

(i) $V = \left\{\begin{pmatrix}x\\y\\z\end{pmatrix} \text{ in } \mathbf{R}^3 \;\middle|\; x - y + z = -1\right\}.$

(ii) The solution set for the homogeneous equation $Ax = 0$, where $A = \begin{pmatrix}1 & -2 & 0\\0 & 0 & 1\end{pmatrix}$.

(iii) The 3-eigenspace of $\begin{pmatrix}3 & -1 & 1\\0 & 3 & 0\\0 & 0 & 3\end{pmatrix}$.

c) (2 points) Suppose A is a 13×15 matrix and the equation $Ax = 0$ has 3 free variables in its solution set. Which **one** of the following describes the column space of A?

(i) $\mathrm{Col}(A)$ is a 10-dimensional subspace of $\mathbf{R}^{13}$.

(ii) $\mathrm{Col}(A)$ is a 10-dimensional subspace of $\mathbf{R}^{15}$.

(iii) $\mathrm{Col}(A)$ is a 12-dimensional subspace of $\mathbf{R}^{13}$.

(iv) $\mathrm{Col}(A)$ is a 12-dimensional subspace of $\mathbf{R}^{15}$.

d) (3 points) Suppose $\{v_1, v_2, v_3\}$ is a linearly independent set of vectors in $\mathbf{R}^n$. Which of the following statements are true? Clearly circle all that apply.

(i) If b is a vector in $\mathbf{R}^n$, then the equation

$$x_1v_1 + x_2v_2 + x_3v_3 = b$$

has at most one solution.

(ii) The vector equation $x_1v_1 + x_2v_2 = 0$ has only the trivial solution $x_1 = x_2 = 0$.

(iii) If $\{v_1, v_2, v_3\}$ is a basis for $\mathbf{R}^n$, then $n = 3$.

Problem 3.

Short answer. You do not need to show your work on this page, and (a)-(c) are unrelated.

a) (2 points) Suppose that $T : \mathbf{R}^n \to \mathbf{R}^m$ is a linear transformation with standard matrix A. Which **one** of the following statements is equivalent to the condition that T is one-to-one? Clearly circle your answer.

(i) For each y in $\mathbf{R}^m$, the matrix equation $Ax = y$ is consistent.

(ii) For each x in $\mathbf{R}^n$, there is at most one y in $\mathbf{R}^m$ so that $T(x) = y$.

(iii) For each y in $\mathbf{R}^m$, there is at most one x in $\mathbf{R}^n$ so that $T(x) = y$.

(iv) A has a pivot in every row.

b) (3 points) Which of the following linear transformations are onto? Clearly circle all that apply.

(i) $T : \mathbf{R}^3 \to \mathbf{R}^2$ given by $T(x, y, z) = (2x + 2y + 2z, x + y + z)$.

(ii) $T : \mathbf{R}^2 \to \mathbf{R}^2$ that rotates vectors by 15° counterclockwise.

(iii) $T : \mathbf{R}^3 \to \mathbf{R}^2$ given by

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & -2 & 2 \\ -1 & 2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

c) (5 points) Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation that reflects vectors across the line $y = x$, and let $U : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation that rotates vectors by 90° **clockwise**.

(i) (2 points) Write the standard matrix A for T. $A = \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix}$

(ii) (2 points) Write the standard matrix B for U. $B = \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix}$

(iii) (1 point) Let $V : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation that first reflects vectors across the line $y = x$, then rotates by 90° clockwise. Which **one** of the following is the standard matrix for V? Clearly circle your answer below.

AB $\qquad\qquad$ BA

Problem 4.

Short answer. You do not need to show your work on this page, and (a)-(d) are unrelated.

a) (2 points) Suppose $\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = 2$. Find $\det\begin{pmatrix} d & e & f \\ 3g-5a & 3h-5b & 3i-5c \\ g & h & i \end{pmatrix}$.

Clearly circle your answer below.

(i) -6 (ii) 6 (iii) 15 (iv) -10 (v) 10

(vi) -30 (vii) 2 (viii) 30 (ix) -15 (x) None of these

b) (2 points) Find the area of the triangle with vertices

$$(2,1), \quad (3,-2), \quad (11,8).$$

Clearly circle your answer below.

(i) 7 (ii) 10 (iii) 17 (iv) 34 (v) 5/2

(vi) 7/2 (vii) 20 (viii) 68 (ix) None of these

c) (4 points) Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation that reflects vectors across the line $y = -8x$, and let A be the standard matrix for T, so $T\begin{pmatrix} x \\ y \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix}$.

(i) In the spaces below, write one eigenvalue λ_1 of A and write one eigenvector v_1 corresponding to λ_1.

$$\lambda_1 = \underline{\qquad\qquad} \qquad v_1 = \begin{pmatrix} \quad \\ \quad \end{pmatrix}.$$

(ii) In the spaces below, write the other eigenvalue λ_2 of A and write one eigenvector v_2 corresponding to λ_2.

$$\lambda_2 = \underline{\qquad\qquad} \qquad v_2 = \begin{pmatrix} \quad \\ \quad \end{pmatrix}.$$

d) (2 points) Which **one** of the following matrices is invertible but not diagonalizable? Clearly circle your answer.

(i) $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ (ii) $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix}$ (iii) $\begin{pmatrix} 2 & 10 & 1 \\ 0 & -2 & 3 \\ 0 & 0 & 1 \end{pmatrix}$ (iv) $\begin{pmatrix} 1 & -3 \\ -3 & 3 \end{pmatrix}$

Problem 5.

Short answer. You do not need to show your work on this page, and (a)-(d) are unrelated.

a) (2 points) Suppose A and B are 3×3 matrices satisfying $\det(A) = -2$ and $\det(B) = 3$. Find $\det(2A^{-1}B)$. Clearly circle your answer below.

(i) -3 (ii) 3 (iii) 6 (iv) -6 (v) 8

(vi) -8 (vii) -12 (viii) 12 (ix) -48 (x) Not enough information

b) (4 points) Suppose that 1, 2, and $3-2i$ are eigenvalues of some $n\times n$ matrix A whose entries are real numbers. Which of the following must be true? Clearly circle all that apply.

(i) A cannot be a 3×3 matrix.

(ii) $3+2i$ must also be an eigenvalue of A.

(iii) If v is an eigenvector of A, then $-v$ is also an eigenvector of A.

(iv) The equation $(A-2I)x = 0$ has only the trivial solution.

c) (2 points) Consider the positive stochastic matrix $A = \begin{pmatrix} 0.6 & 0.5 \\ 0.4 & 0.5 \end{pmatrix}$.

The steady-state vector of A is $\begin{pmatrix} 5/9 \\ 4/9 \end{pmatrix}$. What vector does $A^n \begin{pmatrix} 100 \\ 800 \end{pmatrix}$ approach as n gets very large? Enter your answer in the space below.

$$\begin{pmatrix} \quad \\ \quad \end{pmatrix}$$

d) (2 points) Find A^{-1} if $A = \begin{pmatrix} 9 & 3 \\ 0 & 4 \end{pmatrix}$.

(i) $\frac{1}{36}\begin{pmatrix} 9 & -3 \\ 0 & 4 \end{pmatrix}$ (ii) $\frac{1}{33}\begin{pmatrix} 9 & -3 \\ 0 & 4 \end{pmatrix}$ (iii) $\frac{1}{36}\begin{pmatrix} 4 & 0 \\ -3 & 9 \end{pmatrix}$

(iv) $\frac{1}{33}\begin{pmatrix} 4 & -3 \\ 0 & 9 \end{pmatrix}$ (v) $\frac{1}{36}\begin{pmatrix} 4 & -3 \\ 0 & 9 \end{pmatrix}$ (vi) $\frac{1}{36}\begin{pmatrix} -4 & 3 \\ 0 & -9 \end{pmatrix}$

Problem 6.

Short answer. You do not need to show your work on this page, and (a)-(d) are unrelated.

a) (3 points) Let A be a 7×5 matrix with 3 pivots. Fill in the blanks below.

(i) The dimension of Nul(A^T) is ________.

(ii) $\dim(\text{Row } A) =$ ________.

(iii) $\dim((\text{Row } A)^\perp) =$ ________.

b) (3 points) Suppose W is a subspace of $\mathbf{R}^n$, and let B be the matrix for orthogonal projection onto W. Which of the following are true? Clearly circle all that apply.

(i) If x is in W, then $Bx = x$.

(ii) For each x in $\mathbf{R}^n$, either $Bx = x$ or $Bx = 0$.

(iii) $B^2 = B$.

c) (2 points) Let W be the subspace of $\mathbf{R}^4$ consisting of all vectors $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$ satisfying

$$x_1 - 2x_2 - 3x_3 + x_4 = 0.$$

Which **one** of the following is $W^\perp$?

(i) Span $\left\{ \begin{pmatrix} 1 \\ -2 \\ -3 \\ 1 \end{pmatrix} \right\}$ (ii) Nul$\begin{pmatrix} 1 & -2 & -3 & 1 \end{pmatrix}$ (iii) Col$\begin{pmatrix} 1 & -2 & -3 & 1 \end{pmatrix}$

(iv) Span $\left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ (v) Nul $\begin{pmatrix} 2 & 3 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (vi) Nul $\begin{pmatrix} 2 & 3 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

d) (2 points) For $A = \begin{pmatrix} 1 & -2 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} 0 \\ 9 \\ 5 \end{pmatrix}$, the vector $\widehat{x} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$ is a least-squares solution to the equation $Ax = b$. Which **one** of the following is the closest vector to b in the column space of A? Clearly circle your answer.

(i) $\begin{pmatrix} -3 \\ 4 \end{pmatrix}$ (ii) $\begin{pmatrix} 10 \\ -3 \\ 1 \end{pmatrix}$ (iii) $\begin{pmatrix} 0 \\ 9 \\ 5 \end{pmatrix}$ (iv) $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ (v) $\begin{pmatrix} -1 \\ 5 \\ 14 \end{pmatrix}$ (vi) $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$ (vii) $\begin{pmatrix} -2 \\ 3 \\ 7 \end{pmatrix}$

(viii) None of these

Problem 7.

Free response. Show your work unless otherwise indicated! A correct answer without sufficient work may receive little or no credit.

For this problem, let $A = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 7 & 0 \\ 1 & 2 & 1 \end{pmatrix}$.

a) (2 points) Write the eigenvalues of A. You do not need to show your work on this part.

b) (5 points) For each eigenvalue of A, find a basis for the corresponding eigenspace.

c) (3 points) The matrix A is diagonalizable. Write a 3×3 matrix C and a 3×3 diagonal matrix D so that $A = CDC^{-1}$. Enter your answer below.

$$C = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix} \qquad D = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$$

Problem 8.

Free response. Show your work! A correct answer without sufficient work may receive little or no credit. Parts (a) and (b) are unrelated.

a) (4 points) Let $U : \mathbf{R}^2 \to \mathbf{R}^3$ be the linear transformation satisfying

$$U\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}4\\2\\-6\end{pmatrix}, \qquad U\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}-2\\4\\8\end{pmatrix}.$$

Find the standard matrix A for U. Enter your answer in the space below.

$$A = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$$

b) Let W be the subspace of $\mathbf{R}^3$ given by

$$W = \left\{ \begin{pmatrix}x\\y\\z\end{pmatrix} \text{ in } \mathbf{R}^3 \;\middle|\; x - 6y + 3z = 0 \right\}.$$

(i) (4 points) Find a basis for W.

(ii) (2 points) Is there a matrix A with the property that $\mathrm{Col}(A) = W$? If your answer is yes, write such an A. If your answer is no, justify why there is no such matrix A.

Problem 9.

Free response. Show your work! A correct answer without sufficient work may receive little or no credit.

For this page of the exam, let $W = \text{Span}\left\{\begin{pmatrix}-2\\3\end{pmatrix}\right\}$.

a) (3 points) Find the matrix B for orthogonal projection onto W. In other words, the matrix B so that $Bx = x_W$ for every x in $\mathbf{R}^2$. Enter your answer below.

$$B = \begin{pmatrix} & \\ & \end{pmatrix}.$$

b) Let $x = \begin{pmatrix}8\\1\end{pmatrix}$.

(i) (3 points) Find x_W. In other words, find the orthogonal projection of x onto W. Fully simplify your answer.

(ii) (2 points) Find $x_{W^\perp}$

(iii) (2 points) Find the distance from x to W.

Problem 10.

Use least squares to find the best-fit line $y = Mx + B$ for the data points

$$(0, -5), \qquad (2, 5), \qquad (4, 3).$$

Enter your answer below:

$$y = \underline{\hspace{2em}}x \ + \ \underline{\hspace{2em}}.$$

You must show appropriate work. If you simply guess a line or estimate the equation for the line based on the data points, you will receive little or no credit, even if your answer is correct or nearly correct.

This page is reserved ONLY for work that did not fit elsewhere on the exam.

If you use this page, please clearly indicate (on the problem's page and here) which problems you are doing.