MATH 1553, FINAL EXAM SOLUTIONS
FALL 2023

Name		**GT ID**	

Circle your instructor and lecture below.

Jankowski (A, 8:25-9:15 AM)   Kafer (B, 8:25-9:15 AM)   Irvine (C, 9:30-10:20)

Kafer (D, 9:30-10:20 AM)   He (G, 12:30-1:20 PM)   Goldsztein (H, 12:30-1:20)

Goldsztein (I, 2:00-2:50 PM)   Neto (L, 3:30-4:20 PM)

Yu (M, 3:30-4:20 PM)   Ostrovskii, (N, 5:00-5:50 PM)

Please **read all instructions** carefully before beginning.

- Write your initials at the top of each page. The maximum score on this exam is 100 points, and you have 170 minutes to complete it. Each problem is worth 10 points.
- There are no calculators or aids of any kind (notes, text, etc.) allowed.
- Simplify your answers as much as possible. For example, you may lose points if you do not simplify $\frac{8}{2}$ to 4, or if you do not simplify $\frac{0.1}{0.9}$ to $\frac{1}{9}$, etc.
- As always, RREF means "reduced row echelon form." The "zero vector" in $\mathbf{R}^n$ is the vector in $\mathbf{R}^n$ whose entries are all zero.
- Show your work, unless instructed otherwise. A correct answer without appropriate work will receive little or no credit!
- Unless stated otherwise, **the entries of all matrices on the exam are real numbers**.
- We use $e_1, e_2, \ldots, e_n$ to denote the standard unit coordinate vectors of $\mathbf{R}^n$.
- We will hand out loose scrap paper, but it **will not be graded**. All answers and all work must be written on the exam itself, with no exceptions.
- This exam is double-sided. You should have more than enough space to do every problem on the exam, but if you need extra space, you may use the *back side of the very last page of the exam*. If you do this, you must clearly indicate it.

Please read and sign the following statement.

I, the undersigned, hereby affirm that I will not share the contents of this exam with anyone. Furthermore, I have not received inappropriate assistance in the midst of nor prior to taking this exam. I will not discuss this exam with anyone in any form until after 9:00 PM on Tuesday, December 12.

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Problem 1.

[1 point each]

TRUE or FALSE. Circle **T** if the statement is *always* true. Otherwise, answer **F**. You do not need to show work or justify your answer.

Full solutions are on the next page.

a) **T** **F** If A is a 5×3 matrix, then the columns of A must be linearly dependent.

b) **T** **F** Suppose that u and v are vectors in a subspace W of $\mathbf{R}^n$. Then $2u - 5v$ must also be a vector in W.

c) **T** **F** If $T : \mathbf{R}^5 \to \mathbf{R}^6$ is a linear transformation, then T cannot be onto.

d) **T** **F** If A is a 3×5 matrix and B is a 5×4 matrix, then the linear transformation T given by $T(x) = ABx$ has domain $\mathbf{R}^3$ and codomain $\mathbf{R}^4$.

e) **T** **F** Suppose A is a 4×4 matrix with columns v_1, v_2, v_3, v_4, and suppose

$$v_1 - v_2 + 2v_3 - v_4 = 0.$$

Then A cannot be invertible.

f) **T** **F** Suppose A is a 4×4 matrix with characteristic polynomial

$$\det(A - \lambda I) = (1 - \lambda)^3(7 - \lambda).$$

Then A must be invertible.

g) **T** **F** If u and v are eigenvectors of a 3×3 matrix A, then $u + v$ must also be an eigenvector of A.

h) **T** **F** Let $A = \begin{pmatrix} 4/5 & 1/12 \\ 1/5 & 11/12 \end{pmatrix}$. Then the 1-eigenspace of A is a line in $\mathbf{R}^2$.

i) **T** **F** Suppose W is a subspace in $\mathbf{R}^n$ and u is a vector in W. Then the orthogonal projection of u onto W is the zero vector.

j) **T** **F** Let $W = \operatorname{Span}\left\{ \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} \right\}$, and let B be the matrix for orthogonal projection onto W. Then the null space of B is a line.

Solution.

a) False. For example, $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

b) True. If u and v are in the subspace W, then so is any linear combination of u and v.

c) True. Its standard matrix A is 6×5 and thus has a max of 5 pivots, so it is not possible for A to have a pivot in every row.

d) False: AB is a 3×4 matrix, so the domain of T is $\mathbf{R}^4$ and the codomain of T is $\mathbf{R}^3$.

e) True: the columns of A are linearly dependent as a direct consequence of the fact that $v_1 - v_2 + 2v_3 - v_4 = 0$, so by the Invertible Matrix Theorem, A is not invertible.

f) True: $\det(A) = (1-0)^3(7-0) = 7$, so A is invertible. Alternatively, from the characteristic polynomial of A, we see that the eigenvalues of A are 1 and 7, so 0 is not an eigenvalue of A and therefore A is invertible.

g) False: if u and v are in different eigenspaces, then $u + v$ is never an eigenvector of A. For example, if $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ then e_1 and e_2 are eigenvectors but $A\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ so $e_1 + e_2$ is not an eiggenvector.

h) True: A is positive stochastic so its 1-eigenspace is a line. Alternatively, you could just do computations to find the 1-eigenspace and see it is a line.

i) False: if u is in W then its orthogonal projection onto W is u itself.

j) False: the null space of B is $W^{\perp}$. From the fact that W is a line in $\mathbf{R}^3$ and

$$3 = \dim(W) + \dim(W^{\perp}) = 1 + \dim(W^{\perp}),$$

we get $\dim(W^{\perp}) = 2$, therefore $W^{\perp}$ is a plane.

Problem 2.

Full solutions are on the next page.

a) (2 points) Select the **one** matrix below whose column space and null space satisfy

$$\mathrm{Col}(A) = \mathrm{Span}\left\{\begin{pmatrix}4\\1\end{pmatrix}\right\} \quad \text{and} \quad \mathrm{Nul}(A) = \mathrm{Span}\left\{\begin{pmatrix}-1\\2\end{pmatrix}\right\}.$$

(i) $\begin{pmatrix}4 & -1\\1 & 2\end{pmatrix}$ (ii) $\begin{pmatrix}4 & 8\\1 & 2\end{pmatrix}$ (iii) $\begin{pmatrix}4 & -8\\1 & -2\end{pmatrix}$ [✓ (iv) $\begin{pmatrix}4 & 2\\1 & 1/2\end{pmatrix}$]

(v) $\begin{pmatrix}4 & -2\\1 & -1/2\end{pmatrix}$ (vi) $\begin{pmatrix}4 & 1\\-1 & 2\end{pmatrix}$ (vii) $\begin{pmatrix}4 & -4\\1 & 2\end{pmatrix}$

b) (3 points) Which of the following are subspaces of $\mathbf{R}^3$? Clearly circle all that apply.

(i) $V = \left\{\begin{pmatrix}x\\y\\z\end{pmatrix} \text{ in } \mathbf{R}^3 \;\middle|\; x - y + z = -1\right\}.$

[✓ (ii) The solution set for the homogeneous equation $Ax = 0$, where $A = \begin{pmatrix}1 & -2 & 0\\0 & 0 & 1\end{pmatrix}$.]

[✓ (iii) The 3-eigenspace of $\begin{pmatrix}3 & -1 & 1\\0 & 3 & 0\\0 & 0 & 3\end{pmatrix}$.]

c) (2 points) Suppose A is a 13×15 matrix and the equation $Ax = 0$ has 3 free variables in its solution set. Which **one** of the following describes the column space of A?

(i) Col(A) is a 10-dimensional subspace of $\mathbf{R}^{13}$.

(ii) Col(A) is a 10-dimensional subspace of $\mathbf{R}^{15}$.

[✓ (iii) Col(A) is a 12-dimensional subspace of $\mathbf{R}^{13}$.]

(iv) Col(A) is a 12-dimensional subspace of $\mathbf{R}^{15}$.

d) (3 points) Suppose $\{v_1, v_2, v_3\}$ is a linearly independent set of vectors in $\mathbf{R}^n$. Which of the following statements are true? Clearly circle all that apply.

[✓ (i) If b is a vector in $\mathbf{R}^n$, then the equation$x_1v_1 + x_2v_2 + x_3v_3 = b$has at most one solution.]

[✓ (ii) The vector equation $x_1v_1 + x_2v_2 = 0$ has only the trivial solution $x_1 = x_2 = 0$.]

[✓ (iii) If $\{v_1, v_2, v_3\}$ is a basis for $\mathbf{R}^n$, then $n = 3$.]

Solution.

a) The answer is (iv): $A = \begin{pmatrix} 4 & 2 \\ 1 & 1/2 \end{pmatrix}$. Both its columns are nonzero multiples of $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$ so $\mathrm{Col}(A) = \mathrm{Span}\left\{\begin{pmatrix} 4 \\ 1 \end{pmatrix}\right\}$. Also, one can find its null space directly, or alternatively one can easily see $\dim(\mathrm{Nul}\ A) = 1$ from the Rank Theorem and compute

$$A\begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ 1 & 1/2 \end{pmatrix}\begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

b) (ii) and (iii) are subspaces of $\mathbf{R}^3$. Note (i) is not a subspace of $\mathbf{R}^3$ since V does not even contain the zero vector, as $0 - 0 + 0 \neq -1$.
Part (ii) is a subspace of $\mathbf{R}^3$ since it is the null space of a 2×3 matrix which is automatically a subspace of $\mathbf{R}^3$ no matter what the matrix is. Part (iii) is an eigenspace corresponding to an eigenvalue of a 3×3 matrix and is therefore automatically a subspace of $\mathbf{R}^3$.

c) This is a direct application of the Rank Theorem. We are told $\dim(\mathrm{Nul}\ A) = 3$, so

$$15 = \dim(\mathrm{Nul}\ A) + \dim(\mathrm{Col} A) = 3 + \dim(\mathrm{Col}\ A),$$

therefore $\dim(\mathrm{Col} A) = 12$. Also, $\mathrm{Col}(A)$ is a subspace of $\mathbf{R}^{13}$ since every column of A has exactly 13 entries.

d) (i) Is true: if A is the matrix with columns v_1 through v_3, then $T(x) = Ax$ is a one-to-one linear transformation since $\{v_1, v_2, v_3\}$ is linearly independent, which is precisely the same as saying $Ax = b$ (i.e. the equation $x_1v_1 + x_2v_2 + x_3v_3 = b$) has at most one solution if b is in $\mathbf{R}^n$.

(ii) is true: it is just the statement that $\{v_1, v_2\}$ is a linearly independent set. if $\{v_1, v_2\}$ were not linearly independent, then our larger set $\{v_1, v_2, v_3\}$ could not be linearly independent in the first place.

(iii) is true: if $\{v_1, v_2, v_3\}$ is linearly independent, then $n \geq 3$. If $\{v_1, v_2, v_3\}$ spans $\mathbf{R}^n$, then $n \leq 3$. Here both are true since $\{v_1, v_2, v_3\}$ is a basis for $\mathbf{R}^n$, therefore therefore $n = 3$.

Problem 3.

Full solutions are on the next page.

a) (2 points) Suppose that $T : \mathbf{R}^n \to \mathbf{R}^m$ is a linear transformation with standard matrix A. Which **one** of the following statements is equivalent to the condition that T is one-to-one? Clearly circle your answer.

(i) For each y in $\mathbf{R}^m$, the matrix equation $Ax = y$ is consistent.

(ii) For each x in $\mathbf{R}^n$, there is at most one y in $\mathbf{R}^m$ so that $T(x) = y$.

[✓ (iii) For each y in $\mathbf{R}^m$, there is at most one x in $\mathbf{R}^n$ so that $T(x) = y$.]

(iv) A has a pivot in every row.

b) (3 points) Which of the following linear transformations are onto? Clearly circle all that apply.

(i) $T : \mathbf{R}^3 \to \mathbf{R}^2$ given by $T(x, y, z) = (2x + 2y + 2z, x + y + z)$.

[✓ (ii)] $T : \mathbf{R}^2 \to \mathbf{R}^2$ that rotates vectors by 15° counterclockwise.

[✓ (iii)] $T : \mathbf{R}^3 \to \mathbf{R}^2$ given by

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & -2 & 2 \\ -1 & 2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

c) (5 points) Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation that reflects vectors across the line $y = x$, and let $U : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation that rotates vectors by 90° **clockwise**.

(i) (2 points) Write the standard matrix A for T. $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

(ii) (2 points) Write the standard matrix B for U. $B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

(iii) (1 point) Let $V : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation that first reflects vectors across the line $y = x$, then rotates by 90° clockwise. Which **one** of the following is the standard matrix for V? Clearly circle your answer below.

AB [✓ BA]

Solution.

a) Statement (iii) is the definition of one-to-one. Parts (i) and (iv) are equivalent to “onto” and part (ii) is just the statement that T is a transformation.

b) (ii) and (iii) are onto, but (i) is not.

For (i), we could either compute A and discover it has only one pivot but two rows, or we could just observe that every vector in the range of T is in the span of $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ since the first entry is twice the second entry for every vector in the range. Either way, we see that the range of T is just a line in $\mathbf{R}^2$, so T is not onto.

Part (ii) is onto, since T is invertible and its inverse is clockwise rotation by 15°.

Part (iii) is onto because one row operation on $\begin{pmatrix} 1 & -2 & 2 \\ -1 & 2 & 2 \end{pmatrix}$ gives $\begin{pmatrix} 1 & -2 & 2 \\ 0 & 0 & 4 \end{pmatrix}$, so the matrix has a pivot in every row.

c) For (i) and (ii), these are standard matrices, but if you don’t remember their formulas then you can quickly derive them by computing

$$A = \begin{pmatrix} T\begin{pmatrix} 1 \\ 0 \end{pmatrix} & T\begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} U\begin{pmatrix} 1 \\ 0 \end{pmatrix} & U\begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix}.$$

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

For (iii), the order of operations specifies that if we take $V(z)$, then first we reflect z by taking Az, and THEN we rotate by multiplying the result by B, therefore $V(z) = BAz$.

Problem 4.

Full solutions are on the next page.

a) (2 points) Suppose $\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = 2$. Find $\det\begin{pmatrix} d & e & f \\ 3g-5a & 3h-5b & 3i-5c \\ g & h & i \end{pmatrix}$.

Clearly circle your answer below.

(i) -6 (ii) 6 (iii) 15 (iv) -10 [✓ (v) 10]

(vi) -30 (vii) 2 (viii) 30 (ix) -15 (x) None of these

b) (2 points) Find the area of the triangle with vertices

$$(2,1), \quad (3,-2), \quad (11,8).$$

Clearly circle your answer below.

(i) 7 (ii) 10 [✓ (iii) 17] (iv) 34 (v) $5/2$

(vi) $7/2$ (vii) 20 (viii) 68 (ix) None of these

c) (4 points) Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation that reflects vectors across the line $y = -8x$, and let A be the standard matrix for T, so $T\begin{pmatrix} x \\ y \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix}$.

(i) In the spaces below, write one eigenvalue λ_1 of A and write one eigenvector v_1 corresponding to λ_1.

$$\boxed{\lambda_1 = 1 \qquad v_1 = \begin{pmatrix} 1 \\ -8 \end{pmatrix}}.$$

(ii) In the spaces below, write the other eigenvalue λ_2 of A and write one eigenvector v_2 corresponding to λ_2.

$$\boxed{\lambda_2 = -1 \qquad v_2 = \begin{pmatrix} 8 \\ 1 \end{pmatrix}}.$$

d) (2 points) Which **one** of the following matrices is invertible but not diagonalizable? Clearly circle your answer.

(i) $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ [✓ (ii) $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix}$] (iii) $\begin{pmatrix} 2 & 10 & 1 \\ 0 & -2 & 3 \\ 0 & 0 & 1 \end{pmatrix}$ (iv) $\begin{pmatrix} 1 & -3 \\ -3 & 3 \end{pmatrix}$

Solution.

a) The sequence of row operations to get from beginning to end is to do one row swap (multiply det by -1), one row scale by a factor of -5 (multiply det by -5), and one row-replacement (no change in det), so the final determinant is

$$2(-1)(-5) = 10.$$

The sequence of steps is shown below.

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} d & e & f \\ a & b & c \\ g & h & i \end{pmatrix} \xrightarrow{R_2=-5R_2} \begin{pmatrix} d & e & f \\ -5a & -5b & -5c \\ g & h & i \end{pmatrix}$$

$$\xrightarrow{R_2=R_2+3R_3} \begin{pmatrix} d & e & f \\ 3g-5a & 3h-5b & 3i-5c \\ g & h & i \end{pmatrix}.$$

b) The vector from $(2,1)$ to $(3,-2)$ is $(1,-3)$, and the vector from $(2,1)$ to $(11,8)$ is $(9,7)$. Therefore, the area is

$$\frac{1}{2}\left|\det\begin{pmatrix} 1 & 9 \\ -3 & 7 \end{pmatrix}\right| = \frac{1}{2}(7+27) = 17.$$

c) The eigenvalues of the reflection are 1 and -1.

The 1-eigenspace is the line $y = -8x$, so an eigenvector is $v_1 = \begin{pmatrix} 1 \\ -8 \end{pmatrix}$ or any nonzero scalar multiple of it.

The (-1)-eigenspace is the line through the origin perpendicular to $y = -8x$, which is the line $y = x/8$, therefore a corresponding eigenvector is $v_2 = \begin{pmatrix} 8 \\ 1 \end{pmatrix}$ or $v_2 = \begin{pmatrix} 1 \\ 1/8 \end{pmatrix}$ or any nonzero scalar multiple of this vector.

d) Matrix (i) is not invertible. Matrix (iii) is diagonalizable because it is 3 × 3 with 3 different real eigenvalues, and matrix (iv) is diagonalizable because it is a 2 × 2 matrix with 2 different real eigenvalues.

Matrix (ii) is invertible but not diagonalizable. Note $\lambda = 1$ has algebraic multiplicity 2, but $A-I = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{pmatrix}$ which has two pivots and thus $\mathrm{Nul}(A-I)$ is 1-dimensional. Therefore, the geometric multiplicity of $\lambda = 1$ is less than its algebraic multiplicity, so A is not diagonalizable.

Problem 5.

Full solutions are on the next page.

a) (2 points) Suppose A and B are 3×3 matrices satisfying $\det(A) = -2$ and $\det(B) = 3$. Find $\det(2A^{-1}B)$. Clearly circle your answer below.

(i) -3 (ii) 3 (iii) 6 (iv) -6 (v) 8

(vi) -8 [✓ (vii) -12] (viii) 12 (ix) -48 (x) Not enough information

b) (4 points) Suppose that 1, 2, and $3-2i$ are eigenvalues of some $n \times n$ matrix A whose entries are real numbers. Which of the following must be true? Clearly circle all that apply.

[✓ (i) A cannot be a 3×3 matrix.]

[✓ (ii) $3+2i$ must also be an eigenvalue of A.]

[✓ (iii) If v is an eigenvector of A, then $-v$ is also an eigenvector of A.]

(iv) The equation $(A-2I)x = 0$ has only the trivial solution.

c) (2 points) Consider the positive stochastic matrix $A = \begin{pmatrix} 0.6 & 0.5 \\ 0.4 & 0.5 \end{pmatrix}$.

The steady-state vector of A is $\begin{pmatrix} 5/9 \\ 4/9 \end{pmatrix}$. What vector does $A^n \begin{pmatrix} 100 \\ 800 \end{pmatrix}$ approach as n gets very large? Enter your answer in the space below.

$$\boxed{\begin{pmatrix} 500 \\ 400 \end{pmatrix}}$$

d) (2 points) Find A^{-1} if $A = \begin{pmatrix} 9 & 3 \\ 0 & 4 \end{pmatrix}$.

(i) $\frac{1}{36}\begin{pmatrix} 9 & -3 \\ 0 & 4 \end{pmatrix}$ (ii) $\frac{1}{33}\begin{pmatrix} 9 & -3 \\ 0 & 4 \end{pmatrix}$ (iii) $\frac{1}{36}\begin{pmatrix} 4 & 0 \\ -3 & 9 \end{pmatrix}$

(iv) $\frac{1}{33}\begin{pmatrix} 4 & -3 \\ 0 & 9 \end{pmatrix}$ [✓ (v) $\frac{1}{36}\begin{pmatrix} 4 & -3 \\ 0 & 9 \end{pmatrix}$] (vi) $\frac{1}{36}\begin{pmatrix} -4 & 3 \\ 0 & -9 \end{pmatrix}$

Solution.

a) $\det(2A^{-1}B) = 2^3 \det(A^{-1}B) = 8 \cdot \dfrac{1}{\det(A)} \cdot \det(B) = 8 \cdot \dfrac{-1}{2} \cdot 3 = -12.$

b) (i) is true: $3+2i$ must also be an eigenvalue of A, therefore A has at least 4 different eigenvalues which is only possible if A is 4×4 or larger.

(ii) is true because if λ is an eigenvalue of a real matrix, then so is $\overline{\lambda}$. Here, $3-2i$ is an eigenvalue, therefore its complex conjugate $3+2i$ must also be an eigenvalue.

(iii) is true because eigenspaces are subspaces, therefore if v is an eigenvector then so is the nonzero vector $-v$.

(iv) is false: 2 is an eigenvalue of A, so $A-2I$ is not invertible, therefore $(A-2I)x = 0$ has infinitely many solutions.

c) Since A is positive stochastic, we know $A^n v \to$ (sum of entries in v) $\begin{pmatrix} 5/9 \\ 4/9 \end{pmatrix}$, so

$$A^n \begin{pmatrix} 100 \\ 800 \end{pmatrix} \to 900 \begin{pmatrix} 5/9 \\ 4/9 \end{pmatrix} = \begin{pmatrix} 500 \\ 400 \end{pmatrix}.$$

d) If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ satisfies $ad - bc \neq 0$, then $A^{-1} = \dfrac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

Here, $A = \begin{pmatrix} 9 & 3 \\ 0 & 4 \end{pmatrix}$ so

$$A^{-1} = \frac{1}{36-0} \begin{pmatrix} 4 & -3 \\ 0 & 9 \end{pmatrix} = \frac{1}{36} \begin{pmatrix} 4 & -3 \\ 0 & 9 \end{pmatrix}.$$

Problem 6.

Full solutions are on the next page.

a) (3 points) Let A be a 7×5 matrix with 3 pivots. Fill in the blanks below.

(i) The dimension of Nul(A^T) is $\underline{\quad 4 \quad}$

(ii) $\dim(\text{Row } A) = \underline{\quad 3 \quad}$

(iii) $\dim((\text{Row } A)^{\perp}) = \underline{\quad 2 \quad}$

b) (3 points) Suppose W is a subspace of $\mathbf{R}^n$, and let B be the matrix for orthogonal projection onto W. Which of the following are true? Clearly circle all that apply.

[✓ (i)] If x is in W, then $Bx = x$.

(ii) For each x in $\mathbf{R}^n$, either $Bx = x$ or $Bx = 0$.

[✓ (iii)] $B^2 = B$.

c) (2 points) Let W be the subspace of $\mathbf{R}^4$ consisting of all vectors $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$ satisfying

$x_1 - 2x_2 - 3x_3 + x_4 = 0$.
Which **one** of the following is $W^{\perp}$?

[✓ (i) Span $\left\{ \begin{pmatrix} 1 \\ -2 \\ -3 \\ 1 \end{pmatrix} \right\}$] (ii) Nul$\begin{pmatrix} 1 & -2 & -3 & 1 \end{pmatrix}$ (iii) Col$\begin{pmatrix} 1 & -2 & -3 & 1 \end{pmatrix}$

(iv) Span $\left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ (v) Nul $\begin{pmatrix} 2 & 3 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (vi) Nul $\begin{pmatrix} 2 & 3 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

d) (2 points) For $A = \begin{pmatrix} 1 & -2 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} 0 \\ 9 \\ 5 \end{pmatrix}$, the vector $\widehat{x} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$ is a least-squares solution to the equation $Ax = b$. Which **one** of the following is the closest vector to b in the column space of A? Clearly circle your answer.

(i) $\begin{pmatrix} -3 \\ 4 \end{pmatrix}$ (ii) $\begin{pmatrix} 10 \\ -3 \\ 1 \end{pmatrix}$ (iii) $\begin{pmatrix} 0 \\ 9 \\ 5 \end{pmatrix}$ (iv) $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ (v) $\begin{pmatrix} -1 \\ 5 \\ 14 \end{pmatrix}$ (vi) $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$ [✓ (vii) $\begin{pmatrix} -2 \\ 3 \\ 7 \end{pmatrix}$]

Solution.

a) Since A is 7×5 with 3 pivots, we know that $\dim(\text{Col } A) = 3$, so $\text{Col}(A)$ is a 3-dimensional subspace of $\mathbf{R}^7$. We will also use the fact that if W is a subspace of $\mathbf{R}^n$ then $\dim(W) + \dim(W^\perp) = n$.

(i) $(\text{Col } A)^\perp = \text{Nul}(A^T)$, so

$$7 = \dim(\text{Col } A) + \dim(\text{Nul } A^T) = 3 + \dim(\text{Nul } A^T),$$

therefore the dimension of $\text{Nul}(A^T)$ is 4.

(ii) $\dim(\text{Row } A) = \dim(\text{Col } A)$ which is 3 since A has 3 pivots.

(iii) $5 = \dim(\text{Row } A) + \dim((\text{Row} A)^\perp) = 3 + \dim((\text{Row} A)^\perp)$ so $\dim(\text{Row } A)^\perp = 2$.

b) (i) is true by the definition of orthogonal projection.

(ii) is false since many vectors are not in W or $W^\perp$. For example, if B is orthogonal projection onto the x_1-axis in $\mathbf{R}^2$, then

$$B\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$$

which is neither $\begin{pmatrix}1\\1\end{pmatrix}$ nor $\begin{pmatrix}0\\0\end{pmatrix}$.

(iii) is true from one of our standard properties of orthogonal projections.

c) $W = \text{Nul}(A)$ for $A = \begin{pmatrix}1 & -2 & -3 & 1\end{pmatrix}$, so

$$W^\perp = (\text{Nul } A)^\perp = \text{Row}(A) = \text{Span}\left\{\begin{pmatrix}1\\-2\\-3\\1\end{pmatrix}\right\}.$$

d) The closest vector to b in $\text{Col}(A)$ is the orthogonal projection of b onto $\text{Col}(A)$:

$$b_{\text{Col}(A)} = A\widehat{x} = \begin{pmatrix}1 & -2\\0 & 1\\1 & 1\end{pmatrix}\begin{pmatrix}4\\3\end{pmatrix} = \begin{pmatrix}-2\\3\\7\end{pmatrix}.$$

Problem 7.

Free response. Show your work unless otherwise indicated! A correct answer without sufficient work may receive little or no credit.

For this problem, let $A = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 7 & 0 \\ 1 & 2 & 1 \end{pmatrix}$.

a) (2 points) Write the eigenvalues of A. You do not need to show your work on this part.
A is lower-triangular, so its eigenvalues are its diagonal entries: $\lambda = 1$ and $\lambda = 7$.

b) (5 points) For each eigenvalue of A, find a basis for the corresponding eigenspace.

For the 1-eigenspace:

$$(A - I \ 0) = \left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 3 & 6 & 0 & 0 \\ 1 & 2 & 0 & 0 \end{array}\right) \xrightarrow[\text{then } R_2=R_2-3R_1]{R_1\leftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

This gives $x_1 + 2x_2 = 0$ with x_2 and x_3 free, so

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2x_2 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \quad \text{Basis for 1-eig} : \left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

For the 7-eigenspace:

$$(A - 7I \ 0) = \left(\begin{array}{ccc|c} -6 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 1 & 2 & -6 & 0 \end{array}\right) \xrightarrow[\text{then } R_1=-R_1/6]{\text{destroy entries below -6}} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & -6 & 0 \end{array}\right) \xrightarrow[\text{then } R_2\leftrightarrow R_3]{R_3=R_3/2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

This gives $x_1 = 0$ with $x_2 - 3x_3 = 0$, so $x_2 = 3x_3$ and x_3 is free.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 3x_3 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}. \quad \text{Basis for 7-eig} : \left\{ \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} \right\}.$$

c) (3 points) The matrix A is diagonalizable. Write a 3×3 matrix C and a 3×3 diagonal matrix D so that $A = CDC^{-1}$. Enter your answer below.

We form C using linearly independent eigenvectors and form D using the eigenvalues written **in the corresponding order**. Many answers are possible. For example,

$$C = \begin{pmatrix} -2 & 0 & 0 \\ 1 & 0 & 3 \\ 0 & 1 & 1 \end{pmatrix}, \qquad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 7 \end{pmatrix}$$

or

$$C = \begin{pmatrix} 0 & -2 & 0 \\ 3 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \qquad D = \begin{pmatrix} 7 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Problem 8.

Free response. Show your work! A correct answer without sufficient work may receive little or no credit. Parts (a) and (b) are unrelated.

a) (4 points) Let $U : \mathbf{R}^2 \to \mathbf{R}^3$ be the linear transformation satisfying

$$U\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}4\\2\\-6\end{pmatrix}, \qquad U\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}-2\\4\\8\end{pmatrix}.$$

Find the standard matrix A for U. Enter your answer in the space below.

By linearity properties, we have

$$U\begin{pmatrix}1\\0\end{pmatrix} + U\begin{pmatrix}0\\2\end{pmatrix} = U\begin{pmatrix}1\\2\end{pmatrix}, \qquad U\begin{pmatrix}1\\0\end{pmatrix} + 2U\begin{pmatrix}0\\1\end{pmatrix} = U\begin{pmatrix}1\\2\end{pmatrix},$$

$$\begin{pmatrix}4\\2\\-6\end{pmatrix} + 2U\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}-2\\4\\8\end{pmatrix}, \qquad 2U\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}-6\\2\\14\end{pmatrix}, \qquad U\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}-3\\1\\7\end{pmatrix}.$$

Therefore, $A = \left(U\begin{pmatrix}1\\0\end{pmatrix} \quad U\begin{pmatrix}0\\1\end{pmatrix} \right) = \boxed{\begin{pmatrix}4 & -3\\2 & 1\\-6 & 7\end{pmatrix}}.$

b) Let W be the subspace of $\mathbf{R}^3$ given by

$$W = \left\{ \begin{pmatrix}x\\y\\z\end{pmatrix} \text{ in } \mathbf{R}^3 \;\middle|\; x - 6y + 3z = 0 \right\}.$$

(i) (4 points) Find a basis for W.

$W = \mathrm{Nul}\begin{pmatrix}1 & -6 & 3\end{pmatrix}$, where $\left(\begin{array}{ccc|c}1 & -6 & 3 & 0\end{array}\right)$ gives $x_1 - 6x_2 + 3x_3 = 0$, thus $x_1 = 6x_2 - 3x_3$ and x_2 and x_3 are free.

$$\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}6x_2 - 3x_3\\x_2\\x_3\end{pmatrix} = x_2\begin{pmatrix}6\\1\\0\end{pmatrix} + x_3\begin{pmatrix}-3\\0\\1\end{pmatrix}. \qquad \text{Basis for } W : \left\{ \begin{pmatrix}6\\1\\0\end{pmatrix}, \begin{pmatrix}-3\\0\\1\end{pmatrix} \right\}.$$

Alternatively, we can just find two lin. ind. vectors that satisfy $x - 6y + 3z = 0$ by just plug and chug.

(ii) (2 points) Is there a matrix A with the property that $\mathrm{Col}(A) = W$? If your answer is yes, write such an A. If your answer is no, justify why there is no such matrix A.

Certainly! Just take your basis for W as the columns of A, and in this case $W = \mathrm{Col}\ A$ literally by the definition of column space. $A = \begin{pmatrix}6 & -3\\1 & 0\\0 & 1\end{pmatrix}.$

Problem 9.

Free response. Show your work! A correct answer without sufficient work may receive little or no credit.

For this page of the exam, let $W = \text{Span}\left\{\begin{pmatrix}-2\\3\end{pmatrix}\right\}$.

a) (3 points) Find the matrix B for orthogonal projection onto W. In other words, the matrix B so that $Bx = x_W$ for every x in $\mathbf{R}^2$. Enter your answer below.

With $u = \begin{pmatrix}-2\\3\end{pmatrix}$, we have

$$B = \frac{1}{u\cdot u}uu^T = \frac{1}{(-2)^2+3^2}\begin{pmatrix}-2\\3\end{pmatrix}\begin{pmatrix}-2 & 3\end{pmatrix} = \frac{1}{13}\begin{pmatrix}4 & -6\\-6 & 9\end{pmatrix}.$$

b) Let $x = \begin{pmatrix}8\\1\end{pmatrix}$.

(i) (3 points) Find x_W. In other words, find the orthogonal projection of x onto W. Fully simplify your answer.

$$x_W = Bx = \frac{1}{13}\begin{pmatrix}4 & -6\\-6 & 9\end{pmatrix}\begin{pmatrix}8\\1\end{pmatrix} = \frac{1}{13}\begin{pmatrix}26\\-39\end{pmatrix} = \begin{pmatrix}2\\-3\end{pmatrix}.$$

(ii) (2 points) Find $x_{W^\perp}$

$$x_{W^\perp} = x - x_W = \begin{pmatrix}8\\1\end{pmatrix} - \begin{pmatrix}2\\-3\end{pmatrix} = \begin{pmatrix}6\\4\end{pmatrix}.$$

(iii) (2 points) Find the distance from x to W.

The distance from x to W is $||x_{W^\perp}||$:

$$||x_{W^\perp}|| = \sqrt{6^2+4^2} = \sqrt{36+16} = \sqrt{52} = 2\sqrt{13}.$$

It is fine if the student does not simplify $\sqrt{52}$ to $2\sqrt{13}$ here.

Problem 10.

Use least squares to find the best-fit line $y = Mx + B$ for the data points

$$(0,-5), \qquad (2,5), \qquad (4,3).$$

Enter your answer below:

$$y = \underline{\hspace{2em}}x \ + \ \underline{\hspace{2em}}.$$

You must show appropriate work. If you simply guess a line or estimate the equation for the line based on the data points, you will receive little or no credit, even if your answer is correct or nearly correct.

No line goes through all three points. The corresponding (inconsistent) system is

$$-5 = M(0) + B$$

$$5 = M(2) + B$$

$$3 = M(4) + B$$

and the corresponding matrix equation is $A\begin{pmatrix} M \\ B \end{pmatrix} = b$ where $A = \begin{pmatrix} 0 & 1 \\ 2 & 1 \\ 4 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} -5 \\ 5 \\ 3 \end{pmatrix}$.

We solve $A^T A\widehat{x} = A^T b$.

$$A^T A = \begin{pmatrix} 0 & 2 & 4 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 2 & 1 \\ 4 & 1 \end{pmatrix} = \begin{pmatrix} 20 & 6 \\ 6 & 3 \end{pmatrix}, \qquad A^T b = \begin{pmatrix} 0 & 2 & 4 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} -5 \\ 5 \\ 3 \end{pmatrix} = \begin{pmatrix} 22 \\ 3 \end{pmatrix}.$$

$$\left(A^T A \,\middle|\, A^T b\right) = \left(\begin{array}{cc|c} 20 & 6 & 22 \\ 6 & 3 & 3 \end{array}\right) \xrightarrow[\text{then } R_1 = R_1/6]{R_1 \leftrightarrow R_2} \left(\begin{array}{cc|c} 1 & 1/2 & 1/2 \\ 20 & 6 & 22 \end{array}\right) \xrightarrow{R_2 = R_2 - 20R_1} \left(\begin{array}{cc|c} 1 & 1/2 & 1/2 \\ 0 & -4 & 12 \end{array}\right)$$

$$\xrightarrow{R_2 = -R_2/3} \left(\begin{array}{cc|c} 1 & 1/2 & 1/2 \\ 0 & 1 & -3 \end{array}\right) \xrightarrow{R_1 = R_1 - (1/2)R_2} \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -3 \end{array}\right).$$

Thus $\widehat{x} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$. The line is

$$y = 2x - 3.$$

This page is reserved ONLY for work that did not fit elsewhere on the exam.

If you use this page, please clearly indicate (on the problem's page and here) which problems you are doing.