Math 1553 Final Exam, Spring 2025

Name		GT ID	

Circle your instructor and lecture below. Be sure to circle the correct choice!

Jankowski (A+HP, 8:25-9:15 AM)   Jankowski (C, 9:30-10:20 AM)

Al Ahmadieh (I, 2:00-2:50 PM)   Al Ahmadieh (M, 3:30-4:20 PM)

Please read the following instructions carefully.

- Write your initials at the top of each page. The maximum score on this exam is 100 points, and you have 170 minutes to complete it. Each problem is worth 10 points.
- There are no calculators or aids of any kind (notes, text, etc.) allowed. Unless stated otherwise, **the entries of all matrices on the exam are real numbers**.
- Simplify all fractions as much as possible. As always, RREF means "reduced row echelon form." The "zero vector" in $\mathbf{R}^n$ is the vector in $\mathbf{R}^n$ whose entries are all zero.
- On free response problems, show your work, unless instructed otherwise. A correct answer without appropriate work may receive little or no credit!
- We will hand out loose scrap paper, but it **will not be graded** under any circumstances. All answers and work must be written on the exam itself, with no exceptions.
- This exam is double-sided. You should have more than enough space to do every problem on the exam, but if you need extra space, you may use the *back side of the very last page of the exam*. If you do this, you must clearly indicate it.
- You may cite any theorem proved in class or in the sections we covered in the text.
- For questions with bubbles, either fill in the bubble completely or leave it blank. **Do not** mark any bubble with "X" or "/" or any such intermediate marking. Anything other than a blank or filled bubble may result in a 0 on the problem, and regrade requests may be rejected without consideration.

I affirm that I will not share the contents of this exam with anyone. Furthermore, I have not received inappropriate assistance in the midst of nor prior to taking this exam. I will not discuss this exam with anyone in any form until 8:50 PM on Tuesday, April 29.

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1. (2 pts each) TRUE or FALSE. If the statement is **ever** false, fill in the bubble for false. You do not need to show work or justify your answer.

   (a) Suppose W is a 2-dimensional subspace of $\mathbf{R}^4$.

   If $\begin{pmatrix} 1 \\ -2 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 0 \\ 5 \end{pmatrix}$ are in W, then $\left\{ \begin{pmatrix} 1 \\ -2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 5 \end{pmatrix} \right\}$ must be a basis for W.

   ○ True

   ○ False

   (b) If v is an eigenvector of an $n \times n$ matrix A, then $9v$ must also be an eigenvector of A.

   ○ True

   ○ False

   (c) If $T : \mathbf{R}^a \to \mathbf{R}^b$ is a linear transformation that is onto, then $a \leq b$.

   ○ True

   ○ False

   (d) If A is an invertible $n \times n$ matrix, then $\lambda = 0$ cannot be an eigenvalue of A.

   ○ True

   ○ False

   (e) Suppose W is a subspace of $\mathbf{R}^n$ and x is a vector in $\mathbf{R}^n$. If x_W is the orthogonal projection of x onto W, then $x \cdot x_W = 0$.

   ○ True

   ○ False

2. Multiple choice. You do not need to show your work on this page, and there is no partial credit. Parts (a) through (d) are unrelated.

(a) (3 points) Let $V = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \text{ in } \mathbf{R}^2 \;\middle|\; xy = 0 \right\}$. Determine which properties of a subspace are satisfied by V. Clearly fill in the bubble for all that apply.

○ V contains the zero vector of $\mathbf{R}^2$.

○ V is closed under addition. In other words, if u and v are vectors in V, then $u + v$ must be in V.

○ V closed under scalar multiplication? In other words, if u is a vector in V and c is a real number, then cu must be in V.

(b) (2 points) Which **one** of these is a 3-dimensional subspace of $\mathbf{R}^4$? Fill in the bubble for all that apply.

○ $W = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \text{ in } \mathbf{R}^4 \;\middle|\; x - y + z - w = 1 \right\}$

○ $\mathrm{Nul}\begin{pmatrix} 1 & -3 & 4 & 9 \end{pmatrix}$

○ $W = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ -2 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 2 \\ 3 \end{pmatrix} \right\}$

○ $\mathrm{Row}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

(c) (3 points) Suppose A is a 60×80 matrix whose RREF has exactly 35 pivots. Which of the following statements are true?

○ The row space of A is a 35-dimensional subspace of $\mathbf{R}^{80}$.

○ The null space of A is a 45-dimensional subspace of $\mathbf{R}^{60}$.

○ Let $W = \mathrm{Col}(A)$, and let B be the matrix for orthogonal projection onto W. Then the rank of B is 35.

(d) (2 points) Let $T : \mathbf{R}^3 \to \mathbf{R}^3$ be the linear transformation that satisfies

$$T\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}, \quad T\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \text{ and } \quad T\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \quad \text{Find } T\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}.$$

Fill in the bubble for your answer below.

○ $\begin{pmatrix} 0 \\ 2 \\ 4 \end{pmatrix}$ ○ $\begin{pmatrix} -1 \\ 6 \\ 10 \end{pmatrix}$ ○ $\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$ ○ $\begin{pmatrix} -3 \\ 6 \\ 10 \end{pmatrix}$ ○ not enough info

3. Multiple choice. You do not need to show your work on this page, and there is no partial credit. Parts (a) through (d) are unrelated.

(a) (3 points) Let A be an $n \times n$ matrix. Which of the following statements are true? Fill in the bubble for all that apply.

○ If there is an $n \times n$ matrix B so that $AB = I$, then A must be invertible.

○ If the equation $Ax = b$ is inconsistent for some b in $\mathbf{R}^n$, then the homogeneous equation $Ax = 0$ must have infinitely many solutions.

○ If $\lambda = 1$ is an eigenvalue of A, then $\mathrm{Nul}(A - I) = \{0\}$.

(b) (3 points) Let $T : \mathbf{R}^n \to \mathbf{R}^m$ be a linear transformation with standard matrix A, so $T(x) = Ax$. Which of the following statements guarantee that T is onto? Fill in the bubble for all that apply.

○ For every y in $\mathbf{R}^m$, there is at least one x in $\mathbf{R}^n$ so that $T(x) = y$.

○ For every x in $\mathbf{R}^n$, there is at least one y in $\mathbf{R}^m$ so that $T(x) = y$.

○ The equation $Ax = b$ is consistent for each b in $\mathbf{R}^m$.

(c) (2 points) Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation that first rotates vectors by 90 degrees clockwise, then reflects vectors across the line $y = x$.

Find $T\begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Fill in the bubble for your answer below.

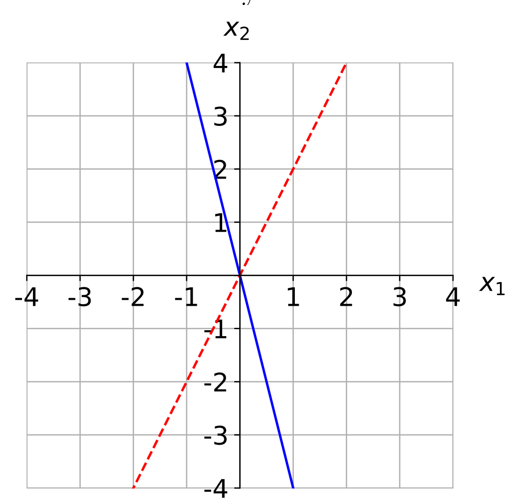
○ $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ ○ $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ ○ $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ ○ $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ ○ $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$

(d) (2 points) Select the matrix whose column space is the solid line and whose null space is the dashed line drawn below. Fill in the bubble for your answer.

○ $\begin{pmatrix} 1 & 1 \\ -4 & 2 \end{pmatrix}$ ○ $\begin{pmatrix} 1 & 1 \\ 2 & -4 \end{pmatrix}$

○ $\begin{pmatrix} 1 & 2 \\ -4 & -8 \end{pmatrix}$ ○ $\begin{pmatrix} 1 & -2 \\ -4 & 8 \end{pmatrix}$

○ $\begin{pmatrix} 1 & 1/2 \\ -4 & -2 \end{pmatrix}$ ○ $\begin{pmatrix} 1 & -1/2 \\ -4 & 2 \end{pmatrix}$



4. Multiple choice. You do not need to show your work on this page, and there is no partial credit. Parts (a) through (d) are unrelated.

(a) (2 points) Find the area of the triangle with vertices $(0,1)$, $(1,3)$, and $(-2,4)$.

○ 1/2 ○ 1 ○ 3/2 ○ 5/2 ○ 7/2

○ 9/2 ○ 3 ○ 5 ○ 9 ○ none of these

(b) (2 points) Suppose A and B are 2×2 matrices satisfying $\det(A) = 2$ and $\det(3AB^{-1}) = 1$. Find $\det(B)$.

○ 3 ○ 6 ○ 9 ○ 12 ○ 18 ○ 1/6

○ 1/9 ○ 1/12 ○ 1/18 ○ not enough info to find $\det(B)$

(c) (2 points) Let A be an 3×3 matrix with characteristic polynomial

$$\det(A - \lambda I) = -\lambda(1 - \lambda)(5 - \lambda).$$

Which **one** of the following statements must be true?

○ A is invertible.

○ Every nonzero vector in $\mathbf{R}^3$ is an eigenvector of A.

○ There is a nonzero vector v in $\mathbf{R}^3$ with the property that v is **not** an eigenvector of A.

○ If u and v are eigenvectors of A, then $u + v$ must also be an eigenvector of A.

(d) (4 points) Suppose A and B are $n \times n$ matrices. Which of the following statements must be true? Clearly fill in the bubble for all that apply.

○ If 0 is an eigenvalue of A, then 0 must also be an eigenvalue of AB.

○ If A and B are invertible, then $(AB)^{-1} = A^{-1}B^{-1}$.

○ If v is a vector in the null space of B, then v must also be in the null space of AB.

○ If w is a vector in the column space of A, then w must also be in the column space of AB.

5. Multiple choice. You do not need to show your work on this page, and there is no partial credit. Parts (a) through (d) are unrelated.

(a) (2 points) Which **one** of the following matrices has no real eigenvalues? Clearly fill in the bubble for your answer.

○ $A = \begin{pmatrix} 1 & -1 \\ 0 & 5 \end{pmatrix}$ ○ $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 1 & 5 \end{pmatrix}$ ○ $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

○ The 2×2 matrix A that reflects vectors across the line $y = -x$, then rotates vectors by $90°$ counterclockwise.

○ The 2×2 matrix A that rotates vectors by $60°$ clockwise.

(b) (2 points) Let $A = \begin{pmatrix} 0.5 & 0.9 \\ 0.5 & 0.1 \end{pmatrix}$. Find the steady-state vector for A.
Fill in the bubble for your answer below.

○ $\begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$ ○ $\begin{pmatrix} 5/9 \\ 4/9 \end{pmatrix}$ ○ $\begin{pmatrix} 9/14 \\ 5/14 \end{pmatrix}$ ○ $\begin{pmatrix} 5/14 \\ 9/14 \end{pmatrix}$ ○ $\begin{pmatrix} 9/10 \\ 1/10 \end{pmatrix}$

○ $\begin{pmatrix} 5/6 \\ 1/6 \end{pmatrix}$ ○ $\begin{pmatrix} 1/6 \\ 5/6 \end{pmatrix}$ ○ $\begin{pmatrix} 1/10 \\ 9/10 \end{pmatrix}$ ○ none of these

(c) (3 points) Let A be the 2×2 matrix that reflects each vector $\begin{pmatrix} x \\ y \end{pmatrix}$ in $\mathbf{R}^2$ across the line $y = 6x$. Which of the following statements are true? Fill in the bubble for all that apply.

○ The eigenvalues of A are -1 and 1.

○ $A\begin{pmatrix} -6 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

○ The 1-eigenspace of A is the span of $\begin{pmatrix} 6 \\ 1 \end{pmatrix}$.

(d) (3 points) Let W be a 4-dimensional subspace of $\mathbf{R}^6$, and let B be the matrix for orthogonal projection onto W. Which of the following statements must be true? Fill in the bubble for all that apply.

○ The eigenvalues of B are -1 and 1.

○ If a vector x is in W and $W^\perp$, then x must be the zero vector.

○ If x is in $\mathbf{R}^6$, then Bx is the closest vector to x in W.

6. Multiple choice. You do not need to show your work on this page, and there is no partial credit. Parts (a) through (d) are unrelated.

(a) (2 points) Find the eigenvalues of the matrix $A = \begin{pmatrix} 4 & 6 \\ -1 & 1 \end{pmatrix}$.

○ 1 and 4 ○ $3 \pm 4i$ ○ $-3 \pm 4i$ ○ $\frac{5}{2} \pm i\frac{\sqrt{15}}{2}$ ○ $2 \pm 3i$

○ $\frac{-5}{2} \pm i\frac{\sqrt{15}}{2}$ ○ $\frac{-3}{2} \pm i\frac{\sqrt{23}}{2}$ ○ $\frac{3}{2} \pm i\frac{\sqrt{23}}{2}$ ○ $\frac{1}{2} \pm i\frac{\sqrt{21}}{2}$

○ none of these

(b) (2 points) Let $W = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$.

Determine which **one** of the following vectors is in $W^{\perp}$. Fill in the bubble for your answer.

○ $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ ○ $\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$ ○ $\begin{pmatrix} 5 \\ -1 \\ 1 \end{pmatrix}$ ○ $\begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$

(c) (4 points) Let A be an $m \times n$ matrix. Which of the following statements must be true? Fill in the bubble for all that apply.

○ If $m > n$, then the matrix transformation $T(x) = Ax$ cannot be one-to-one.

○ $(\text{Col } A)^{\perp} = \text{Nul}(A^T)$

○ $\text{Row } A = (\text{Nul } A)^{\perp}$

○ If the homogeneous equation $Ax = 0$ has only the trivial solution, then $\dim(\text{Col } A) = n$.

(d) (2 points) Let $W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ in } \mathbf{R}^3 \;\middle|\; x - 10y - 2z = 0 \right\}$.

Which **one** of the following describes $W^{\perp}$? Fill in the bubble for your answer.

○ $\text{Span}\left\{ \begin{pmatrix} 1 \\ -10 \\ -2 \end{pmatrix} \right\}$ ○ $\text{Span}\left\{ \begin{pmatrix} 10 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \right\}$ ○ $\text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$

○ $\text{Col}\begin{pmatrix} 1 & -10 & -2 \end{pmatrix}$ ○ $\text{Nul}\begin{pmatrix} 1 \\ -10 \\ -2 \end{pmatrix}$

7. Free response. Show your work unless otherwise indicated! A correct answer without sufficient work may result in little or no credit. Parts (a) and (b) are unrelated.

(a) Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation that reflects vectors in $\mathbf{R}^2$ across the line $y = x$, and let $U : \mathbf{R}^2 \to \mathbf{R}^3$ be the linear transformation given by

$$U(x, y) = (4x - 3y,\ x - 5y,\ x - y).$$

i. (2 points) Find the standard matrix A for T. Enter your answer below.

$$A = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$$

ii. (3 points) Find the standard matrix B for U. Enter your answer below.

$$B = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$$

iii. (3 points) Find the standard matrix C for $U \circ T$. Enter your answer below.

$$C = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$$

(b) (2 points) This part is unrelated to (a).
Is there some $m \times n$ matrix M with the property that

$$\dim((\text{Row } M)^{\perp}) = 1 \quad \text{and} \quad \dim((\text{Col } M)^{\perp}) = 2?$$

If your answer is yes, write such a matrix M in the space below. If your answer is no, fill in the bubble for "no such M exists." You do not need to show your work or justify your answer on this part.

$$M = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$$

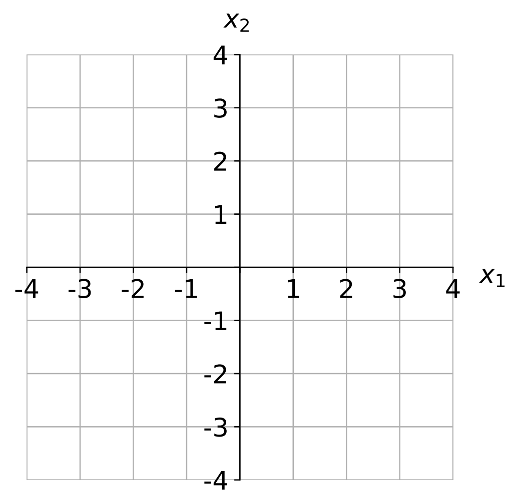
○ no such M exists

8. Free response. Show your work unless otherwise indicated! A correct answer without sufficient work may receive little or no credit.

Consider the matrix $A = \begin{pmatrix} 1 & 4 \\ 1 & -2 \end{pmatrix}$.

(a) Find the eigenvalues of A. Write them here: ________________

(b) For each eigenvalue of A, find a basis for the corresponding eigenspace. Draw and clearly label each eigenspace on the graph below.



(c) A is diagonalizable. In the space provided below, write an invertible matrix C and a diagonal matrix D so that $A = CDC^{-1}$. You do not need to show your work on this part.

$$C = \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix} \qquad D = \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix}$$

9. Free response. Show your work unless otherwise indicated! A correct answer without sufficient work may result in little or no credit.

For this page of the exam, let $W = \text{Span}\left\{\begin{pmatrix}1\\2\end{pmatrix}\right\}$.

(a) i. (2 points) Write one vector v so that the orthogonal projection of v onto W is the zero vector. No work required and no partial credit on this part.

$$v = \begin{pmatrix} \\ \\ \end{pmatrix}$$

ii. (3 points) Find the matrix B for orthogonal projection onto W. In other words, the matrix B so that $Bx = x_W$ for every x in $\mathbf{R}^2$. Enter your answer below.

$$B = \begin{pmatrix} & \\ & \end{pmatrix}.$$

(b) Let $x = \begin{pmatrix}-25\\5\end{pmatrix}$.

(i) (3 points) Find x_W. In other words, find the orthogonal projection of x onto W. Fully simplify your answer.

$$x_W = \begin{pmatrix} \\ \\ \end{pmatrix}$$

(ii) (2 points) Find $x_{W^\perp}$ and write your answer in the space below.

$$x_{W^\perp} = \begin{pmatrix} \\ \\ \end{pmatrix}$$

10. Free response. Show your work! A correct answer without sufficient work may receive little or no credit.

Use least squares to find the best-fit line $y = Mx + B$ for the data points

$$(0,4), \qquad (2,2), \qquad (4,6).$$

Enter your answer below:

$$y = \underline{\hspace{2cm}}x \; + \; \underline{\hspace{2cm}}.$$

You **must** show appropriate work using least squares. If you simply guess a line or estimate the equation for the line based on the data points, you will receive little or no credit, even if your answer is correct or nearly correct.

This page is reserved ONLY for work that did not fit elsewhere on the exam.

If you use this page, please clearly indicate (on the problem's page and here) which problems you are doing.