Math 1553 Final Exam, Fall 2024

Name		GT ID	

Circle your instructor and lecture below. Be sure to circle the correct choice!

Jankowski (A, 8:25-9:15) Wessels(B, 8:25-9:15) Hozumi (C, 9:30-10:20)

Wessels (D, 9:30-10:20) Kim (G, 12:30-1:20) Short (H, 12:30-1:20)

Shubin (I, 2:00-2:50) He (L, 3:30-4:20) Wan (M, 3:30-4:20)

Shubin (N, 5:00-5:50) Denton (W, 8:25-9:15)

Please read the following instructions carefully.

- Write your initials at the top of each page. The maximum score on this exam is 100 points, and you have 170 minutes to complete it. Each problem is worth 10 points.
- There are no calculators or aids of any kind (notes, text, etc.) allowed. Unless stated otherwise, **the entries of all matrices on the exam are real numbers**.
- Simplify all fractions as much as possible. As always, RREF means "reduced row echelon form." The "zero vector" in $\mathbf{R}^n$ is the vector in $\mathbf{R}^n$ whose entries are all zero.
- On free response problems, show your work, unless instructed otherwise. A correct answer without appropriate work may receive little or no credit!
- We will hand out loose scrap paper, but it **will not be graded** under any circumstances. All answers and work must be written on the exam itself, with no exceptions.
- This exam is double-sided. You should have more than enough space to do every problem on the exam, but if you need extra space, you may use the *back side of the very last page of the exam*. If you do this, you must clearly indicate it.
- You may cite any theorem proved in class or in the sections we covered in the text.
- For questions with bubbles, either fill in the bubble completely or leave it blank. **Do not** mark any bubble with "X" or "/" or any such intermediate marking. Anything other than a blank or filled bubble may result in a 0 on the problem, and regrade requests may be rejected without consideration.

I affirm that I will not share the contents of this exam with anyone. Furthermore, I have not received inappropriate assistance in the midst of nor prior to taking this exam. I will not discuss this exam with anyone in any form until 8:50 PM on Tuesday, December 10.

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1. (2 pts each) TRUE or FALSE. If the statement is **ever** false, fill in the bubble for false. You do not need to show work or justify your answer.

   (a) If $\{u, v, w\}$ is a linearly independent set of vectors in $\mathbf{R}^n$, then the set $\{u,\ u-v,\ u-v+2w\}$ must also be linearly independent.

   ○ True ○ False

   (b) If A is an $n \times n$ matrix and the equation $Ax = 0$ has only the trivial solution, then $\det(A) = 0$.

   ○ True ○ False

   (c) If W is a subspace of $\mathbf{R}^n$, then there is at least one matrix A so that $\text{Col}(A) = W$.

   ○ True ○ False

   (d) Let A be a 3×3 matrix with characteristic polynomial

   $$\det(A - \lambda I) = \lambda^2(1-\lambda).$$

   If A is diagonalizable, then the RREF of A must have exactly one pivot.

   ○ True ○ False

   (e) There is a 2×2 positive stochastic matrix that has $-\frac{i}{2}$ as an eigenvalue.

   ○ True ○ False

2. Multiple choice. You do not need to show your work on this page, and there is no partial credit.

(a) (2 points) Consider the following sets in $\mathbf{R}^2$. Clearly fill in the bubble for the **one** set V that satisfies **all** of the following conditions.

(1) V contains the zero vector.

(2) V is *not* closed under addition.

(3) V is closed under scalar multiplication.

○ $V = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \text{ in } \mathbf{R}^2 \;\middle|\; y = |x| \right\}$ ○ $V = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \text{ in } \mathbf{R}^2 \;\middle|\; y \geq 2x \right\}$

○ $V = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \text{ in } \mathbf{R}^2 \;\middle|\; x^2 + y^2 \leq 1 \right\}$ ○ $V = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \text{ in } \mathbf{R}^2 \;\middle|\; xy \leq 0 \right\}$

(b) (2 points) Suppose A is a 70×40 matrix and the solution set to $Ax = 0$ is 25-dimensional. Which **one** of the following describes the column space of A?

○ $\mathrm{Col}(A)$ is a 15-dimensional subspace of $\mathbf{R}^{70}$.

○ $\mathrm{Col}(A)$ is a 15-dimensional subspace of $\mathbf{R}^{40}$.

○ $\mathrm{Col}(A)$ is a 25-dimensional subspace of $\mathbf{R}^{70}$.

○ $\mathrm{Col}(A)$ is a 25-dimensional subspace of $\mathbf{R}^{40}$.

○ $\mathrm{Col}(A)$ is a 45-dimensional subspace of $\mathbf{R}^{70}$.

(c) (4 points) Suppose A is a square matrix with characteristic polynomial

$$\det(A - \lambda I) = -\lambda(10 - \lambda)^2.$$

Which of the following statements are true? Fill in the bubble for all that apply.

○ The null space of A must be a line.

○ Let B be the standard matrix for orthogonal projection onto the column space of A. Then the eigenvalues of B are 0 and 1.

○ A cannot be diagonalizable.

○ A is a 3×3 matrix.

(d) (2 points) Find the value of c so that $\begin{pmatrix} -2 \\ 4 \\ c \end{pmatrix}$ and $\begin{pmatrix} c \\ -5 \\ 1 \end{pmatrix}$ are orthogonal. Clearly fill in the bubble for your answer.

○ $c = \dfrac{20}{3}$ ○ $c = -\dfrac{20}{3}$ ○ $c = 0$ ○ $c = 20$

○ $c = -20$ ○ There is no value of c that makes them orthogonal.

3. Multiple choice. There is no partial credit on this page, and you do not need to show your work.

(a) (2 points) Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation that first reflects vectors across the line $y = -x$, then rotates vectors by $90°$ clockwise. Find $T\begin{pmatrix}0\\4\end{pmatrix}$. Fill in the bubble for your answer below.

○ $\begin{pmatrix}0\\4\end{pmatrix}$ ○ $\begin{pmatrix}0\\-4\end{pmatrix}$ ○ $\begin{pmatrix}-4\\0\end{pmatrix}$ ○ $\begin{pmatrix}-2\\2\end{pmatrix}$ ○ $\begin{pmatrix}2\\-2\end{pmatrix}$

(b) (2 points) Suppose A is a matrix with columns $v_1, \ldots, v_k$. Which **one** of the following is **NOT** true? Clearly fill in the bubble for your one answer.

○ $(\text{Row } A)^{\perp} = \text{Nul}(A)$.

○ $\text{Col}(A) = (\text{Nul } A^T)^{\perp}$.

○ For the subspace $W = \text{Span}\{v_1, \ldots, v_k\}$, we have $W^{\perp} = \text{Nul}(A)$.

○ $\dim(\text{Row } A) = \dim(\text{Col } A)$.

(c) (3 points) Select the **one** matrix A below that satisfies both of the following conditions:

- The only eigenvalue of A is $\lambda = 4$.
- The 4-eigenspace of A is a line.

Fill in the bubble for your answer below.

○ $A = \begin{pmatrix}4 & 1\\1 & 4\end{pmatrix}$ ○ $A = \begin{pmatrix}4 & 1 & 0\\0 & 4 & 0\\0 & 0 & 4\end{pmatrix}$ ○ $A = \begin{pmatrix}4 & 1 & -1 & 2\\0 & 4 & 1 & 0\\0 & 0 & 4 & 0\\0 & 0 & 0 & 4\end{pmatrix}$

○ $A = \begin{pmatrix}4 & -1 & 1 & 0\\0 & 4 & 1 & 0\\0 & 0 & 4 & 2\\0 & 0 & 0 & 4\end{pmatrix}$ ○ $A = \begin{pmatrix}4 & -2 & 0\\0 & 4 & 0\\0 & 0 & 1\end{pmatrix}$ ○ $A = \begin{pmatrix}4 & 0\\0 & 4\end{pmatrix}$

(d) (3 points) Let W be the subspace of $\mathbf{R}^4$ consisting of all vectors $\begin{pmatrix}x\\y\\z\\w\end{pmatrix}$ that satisfy $x - 2y - 3z + w = 0$, and let B be the standard matrix for orthogonal projection onto W. Which of the following are true? Fill in the bubble for all that apply.

○ $\dim(\text{Nul}(B)) = 3$ ○ $Bx = x$ for all x in W ○ $B\begin{pmatrix}1\\-2\\-3\\1\end{pmatrix} = \begin{pmatrix}1\\-2\\-3\\1\end{pmatrix}$

4. Multiple choice. You do not need to show your work, and there is no partial credit.

(a) (3 points) Suppose $\{v_1, v_2, v_3, v_4\}$ is a linearly independent set of vectors in $\mathbf{R}^n$. Which of the following statements are true? Fill in the bubble for all that apply.

- ○ If b is a vector in $\mathbf{R}^n$, then the equation $x_1v_1 + x_2v_2 + x_3v_3 + x_4v_4 = b$ must have exactly one solution.
- ○ The set $\{v_1, v_2, v_3\}$ is linearly independent.
- ○ Suppose A is the matrix whose four columns are v_1, v_2, v_3, and v_4. Then the linear transformation $T(x) = Ax$ must be one-to-one.

(b) (2 points) Let $W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ in } \mathbf{R}^3 \;\middle|\; x - 3y + z = 0 \right\}$. Which **one** of the following is equal to W? Fill in the bubble for your answer.

○ Col $\begin{pmatrix} 3 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$ ○ Nul $\begin{pmatrix} 3 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$ ○ Nul $\begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}$ ○ Span $\left\{ \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} \right\}$

(c) (2 points) Suppose $\det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = 1$. Find $\det \begin{pmatrix} 5a-d & 5b-e & 5c-f \\ a & b & c \\ 3g & 3h & 3i \end{pmatrix}$.

○ 1 ○ -1 ○ 3 ○ -3 ○ 5 ○ -5

○ 10 ○ -10 ○ 15 ○ -15 ○ none of these

(d) (3 points) Abacus Seltzer and Barren Seltzer compete for a market of 100 customers who drink seltzer each day. Today, Abacus Seltzer has 25 customers and Barren Seltzer has 75 customers. Each day:

- 80% of Abacus Seltzer's customers keep drinking Abacus Seltzer, while 20% switch to Barren Seltzer.
- 65% of Barren Seltzer's customers keep drinking Barren Seltzer, while 35% switch to Abacus Seltzer.

Fill in the bubble below that gives a positive stochastic matrix A and a vector x so that Ax will give the number of customers for Abacus Seltzer and Barren Seltzer (in that order) tomorrow.

○ $A = \begin{pmatrix} 0.8 & 0.2 \\ 0.65 & 0.35 \end{pmatrix}$, $x = \begin{pmatrix} 25 \\ 75 \end{pmatrix}$ ○ $A = \begin{pmatrix} 0.8 & 0.35 \\ 0.2 & 0.65 \end{pmatrix}$, $x = \begin{pmatrix} 25 \\ 75 \end{pmatrix}$

○ $A = \begin{pmatrix} 0.8 & 0.65 \\ 0.2 & 0.35 \end{pmatrix}$, $x = \begin{pmatrix} 25 \\ 75 \end{pmatrix}$ ○ $A = \begin{pmatrix} 0.8 & 0.2 \\ 0.35 & 0.65 \end{pmatrix}$, $x = \begin{pmatrix} 25 \\ 75 \end{pmatrix}$

5. Multiple choice. You do not need to show your work on this page, and there is no partial credit.

(a) (2 points) Which **one** of the following matrices A has the property that $\mathrm{Col}(A) = \mathrm{Nul}(A)$? Clearly fill in the bubble for your answer.

○ $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ ○ $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ ○ $A = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$

○ $A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 0 \\ 1 & 1 & -1 \end{pmatrix}$ ○ $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ ○ $A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

(b) (2 points) Find the area of the triangle with vertices

$$(1, 2), \qquad (5, 1), \qquad (6, -4).$$

Fill in the bubble for your answer below.

○ 5 ○ 9 ○ 11 ○ 15 ○ 19 ○ 29

○ $\frac{5}{2}$ ○ $\frac{9}{2}$ ○ $\frac{19}{2}$ ○ $\frac{29}{2}$ ○ none of these

(c) (3 points) Suppose W is a subspace of $\mathbf{R}^n$. Which of the following statements are true? Fill in the bubble for all that apply.

○ If x is a vector in $\mathbf{R}^n$, then x must be in W or $W^\perp$.

○ If x is a vector in $W^\perp$, then the orthogonal projection of x onto W must be the zero vector.

○ If x is a vector in both W and $W^\perp$, then x must be the zero vector.

(d) (3 points) Suppose A is an $m \times n$ matrix and b is a vector in $\mathbf{R}^m$. Which of the following statements are true? Fill in the bubble for all that apply.

○ The equation $Ax = b$ is consistent if and only if b is a linear combination of the columns of A.

○ If w is the orthogonal projection of b onto the column space of A, then the equation $Ax = w$ must be consistent.

○ If $\widehat{x}$ is a least-squares solution to $Ax = b$, then $\widehat{x}$ is the closest vector to b in $\mathrm{Col}(A)$.

6. Multiple choice and short answer. You do not need to show your work on this page, and there is no partial credit except on (d).

(a) (2 points) Suppose $T : \mathbf{R}^k \to \mathbf{R}^\ell$ is a transformation. Which **one** of the following conditions guarantees that T is one-to-one? Fill in the bubble for your answer.

○ For each a in $\mathbf{R}^k$, there is at least one b in $\mathbf{R}^\ell$ so that $T(a) = b$.

○ For each a in $\mathbf{R}^k$, there is at most one b in $\mathbf{R}^\ell$ so that $T(a) = b$.

○ For each b in $\mathbf{R}^\ell$, there is at least one a in $\mathbf{R}^k$ so that $T(a) = b$.

○ For each b in $\mathbf{R}^\ell$, there is at most one a in $\mathbf{R}^k$ so that $T(a) = b$.

(b) (2 points) Let A be the matrix that reflects every vector $\begin{pmatrix} x \\ y \end{pmatrix}$ in $\mathbf{R}^2$ across the line $y = 10x$. Which **one** of the following vectors is an eigenvector corresponding to the eigenvalue $\lambda = -1$? Fill in the bubble for your answer.

○ $\begin{pmatrix} 1 \\ -10 \end{pmatrix}$ ○ $\begin{pmatrix} -10 \\ 1 \end{pmatrix}$ ○ $\begin{pmatrix} 1 \\ 10 \end{pmatrix}$ ○ $\begin{pmatrix} 10 \\ 1 \end{pmatrix}$ ○ $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ ○ $\begin{pmatrix} 5 \\ 5 \end{pmatrix}$

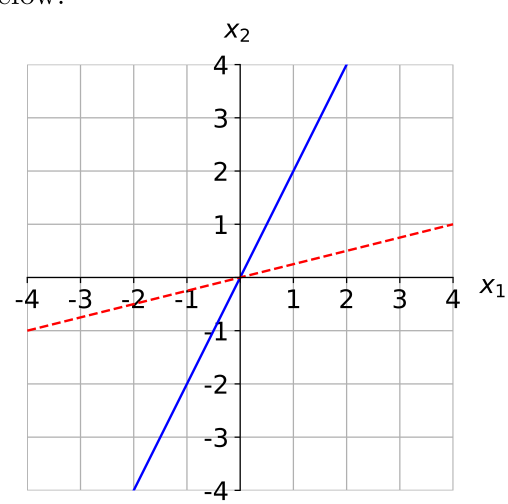
(c) (2 points) Suppose A is a 3×3 matrix whose 2-eigenspace is a plane and whose (-4)-eigenspace is a line. Find $\det(A)$. Fill in the bubble for your answer below.

○ 2 ○ -4 ○ 8 ○ -8 ○ 16

○ -16 ○ 32 ○ -32 ○ not enough info to compute $\det(A)$

(d) (4 points) Write a single matrix A whose column space is the **solid** line below and whose null space is the **dashed** line below.

$$A = \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix}$$



7. Free response. Show your work unless otherwise indicated! A correct answer without sufficient work may result in little or no credit.

Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation that rotates vectors in $\mathbf{R}^2$ by $90°$ **counterclockwise**, and let $U : \mathbf{R}^3 \to \mathbf{R}^2$ be the linear transformation given by

$$U(x, y, z) = (x - 2y + z,\ 3x + y - z).$$

(a) (2 points) Find the standard matrix A for T. Enter your answer below. Evaluate any trigonometric functions. Do not leave your answer in terms of sine and cosine.

$A = \left(\quad \right)$

(b) (3 points) Is there a matrix B so that AB is the 2×2 identity matrix? If your answer is yes, find B and write it in the space below. If your answer is no, fill in the bubble for "no such B exists" and justify your answer.

$B = \left(\quad \right)$ ○ no such B exists

(c) (3 points) Find the standard matrix C for U. Enter your answer below.

$C = \left(\quad \right)$

(d) (2 points) Find the standard matrix M for $T \circ U$. Enter your answer below.

$M = \left(\quad \right)$

8. Free response. This page is worth 10 points. Show your work unless otherwise indicated! A correct answer without sufficient work may receive little or no credit.

Consider the matrix $A = \begin{pmatrix} 4 & 1 & -1 \\ 0 & 0 & 4 \\ 0 & 1 & 3 \end{pmatrix}$.

(a) Find the eigenvalues of A. Enter them here: ________________________.

(b) For each eigenvalue of A, find a basis for the corresponding eigenspace.

(c) A is diagonalizable. In the space provided below, write an invertible matrix C and a diagonal matrix D so that $A = CDC^{-1}$. You do not need to show your work on this part.

$$C = \begin{pmatrix} \quad & \quad & \quad \\ & & \\ & & \end{pmatrix} \qquad D = \begin{pmatrix} \quad & \quad & \quad \\ & & \\ & & \end{pmatrix}$$

9. Free response. Show your work! A correct answer without sufficient work may result in little or no credit.

   (a) (3 points) Let $W = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$. Find a basis for $W^{\perp}$.

   (b) Let $W = \text{Span}\left\{ \begin{pmatrix} 2 \\ -1 \end{pmatrix} \right\}$ and let $y = \begin{pmatrix} 10 \\ 5 \end{pmatrix}$.

      i. (4 pts) Find the closest vector to y in W. Enter your answer here: $\begin{pmatrix} \quad \\ \quad \end{pmatrix}$

      ii. (3 points) Find the distance from y to W. Enter your answer here: __________

10. Free response. This problem is worth 10 points. Show your work!

Use least squares to find the best-fit line $y = Mx + B$ for the data points

$$(1, 3), \qquad (2, 1), \qquad (3, -7).$$

Enter your answer below:

$$y = \underline{\hspace{2cm}}x \; + \; \underline{\hspace{2cm}}.$$

You **must** show appropriate work using least squares. If you simply guess a line or estimate the equation for the line based on the data points, you will receive little or no credit, even if your answer is correct or nearly correct.

This page is reserved ONLY for work that did not fit elsewhere on the exam.

If you use this page, please clearly indicate (on the problem's page and here) which problems you are doing.