Math 1553 Exam 3, SOLUTIONS, Spring 2025

Name		**GT ID**	

Circle your instructor and lecture below. Be sure to circle the correct choice!

Jankowski (A+HP, 8:25-9:15 AM)   Jankowski (C, 9:30-10:20 AM)

Al Ahmadieh (I, 2:00-2:50 PM)   Al Ahmadieh (M, 3:30-4:20 PM)

Please read the following instructions carefully.

- Write your initials at the top of each page. The maximum score on this exam is 70 points, and you have 75 minutes to complete it. Each problem is worth 10 points.
- There are no calculators or aids of any kind (notes, text, etc.) allowed. Unless stated otherwise, **the entries of all matrices on the exam are real numbers**.
- As always, RREF means "reduced row echelon form." The "zero vector" in $\mathbf{R}^n$ is the vector in $\mathbf{R}^n$ whose entries are all zero.
- On free response problems, show your work, unless instructed otherwise. A correct answer without appropriate work may receive little or no credit!
- We will hand out loose scrap paper, but it **will not be graded** under any circumstances. All answers and work must be written on the exam itself, with no exceptions.
- This exam is double-sided. You should have more than enough space to do every problem on the exam, but if you need extra space, you may use the *back side of the very last page of the exam*. If you do this, you must clearly indicate it.
- You may cite any theorem proved in class or in the sections we covered in the text.
- For questions with bubbles, either fill in the bubble completely or leave it blank. **Do not** mark any bubble with "X" or "/" or any such intermediate marking. Anything other than a blank or filled bubble may result in a 0 on the problem, and regrade requests may be rejected without consideration.

I, the undersigned, hereby affirm that I will not share the contents of this exam with anyone. Furthermore, I have not received inappropriate assistance in the midst of nor prior to taking this exam. I will not discuss this exam with anyone in any form until after 7:45 PM on Wednesday, April 9.

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1. TRUE or FALSE. Clearly fill in the bubble for your answer. If the statement is *ever* false, fill in the bubble for False. You do not need to show any work, and there is no partial credit. Each question is worth 2 points.

(a) If A is a 3×3 matrix, then $\det(2A) = 8\det(A)$.

● True

○ False

(b) If A is a 3×3 matrix and $A \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, then $\det(A) = 0$.

● True

○ False

(c) If A is a 3×3 matrix and

$$A \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ 12 \end{pmatrix},$$

then the equation $Ax = 4x$ must have infinitely many solutions.

● True

○ False

(d) If u and v are eigenvectors of an $n \times n$ matrix A, then $u + v$ must also be an eigenvector of A.

○ True

● False

(e) If A is a 3×3 matrix with eigenvalues $\lambda_1 = 0$, $\lambda_2 = 1$, and $\lambda_3 = -4$, then A must be diagonalizable.

● True

○ False

Solution: Problem 1.

(a) True: Since A is 3×3, multiplying A by 2 will multiply each row by 2, which multiplies the determinant by $2^3 = 8$. Therefore, $\det(2A) = 8\det(A)$.

(b) True: the vector $x = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ is a non-trivial solution to the equation $Ax = 0$, so A is not invertible, therefore $\det(A) = 0$.

(c) True: $A\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ 12 \end{pmatrix}$, so the equation $Ax = 4x$ has a non-trivial solution. Therefore, $\lambda = 4$ is an eigenvalue of A, so $\mathrm{Nul}(A - 4I)$ contains infinitely many vectors, which is the same as saying that $Ax = 4x$ has infinitely many solutions.

(d) False: one thing that can go wrong is that u and v are in different eigenspaces, in which case $u+v$ will never be an eigenvector. for example, if $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, then $u = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is an eigenvector of A for $\lambda = 1$, and $v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is an eigenvector for $\lambda = 2$. However, $u + v$ is not an eigenvector of A since

$$A(u+v) = A\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

which is not a scalar multiple of $u + v$.

(e) True, since an $n \times n$ matrix with n distinct real eigenvalues is always diagonalizable. Recall that eigenvalues corresponding to different eigenvectors are automatically linearly independent, so having n different eigenvalues leads to n linearly independent eigenvectors in $\mathbf{R}^n$, therefore the matrix is diagonalizable.

2. **Full solutions are on the next page.**

(a) (2 points) Suppose $\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = 1$. Find $\det\begin{pmatrix} 2a-4d & 2b-4e & 2c-4f \\ a & b & c \\ g & h & i \end{pmatrix}$.

○ 1 ○ −1 ○ 2 ○ −2 ○ 6

● 4 ○ −4 ○ 8 ○ −8 ○ none of these

(b) (2 points) Find the area of the triangle with vertices

$$(1,1), \quad (3,5), \quad (4,-1).$$

○ 1/2 ○ 1 ○ 2 ○ 4 ○ 5 ● 8

○ 10 ○ 16 ○ 64 ○ none of these

(c) (2 points) Let T be the linear transformation $T(x,y) = (3x-y, x-2y)$, and let S be a rectangle in $\mathbf{R}^2$ with area 6. Find the area of $T(S)$.

○ 5 ○ 6 ○ 7 ● 30 ○ 36

○ 42 ○ 48 ○ 72 ○ none of these

(d) (4 points) Suppose A and B are $n \times n$ matrices. Which of the following statements are true? Clearly fill in the bubble for all that apply.

● $\det(AB) = \det(A)\det(B)$.

○ If $\det(A) = 4$, then $\det(A^{-1}) = -4$.

● If $\det(A) = -2$, then $\det(A^4) = 16$.

● If 0 is an eigenvalue of A, then $\det(AB) = 0$.

Solution: Problem 2

(a) To get from the original matrix to the final matrix: we first swap rows 1 and 2, then multiply the new row 1 by -4, then do a row-replacement. The sequence of operations multiplies the determinant by -1, then by -4, then by 1. Therefore the answer is $1(-1)(-4)(1) = 4$.

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} d & e & f \\ a & b & c \\ g & h & i \end{pmatrix} \xrightarrow{R_1 = -4R_1} \begin{pmatrix} -4d & -4e & -4f \\ a & b & c \\ g & h & i \end{pmatrix}$$
$$\xrightarrow{R_1 = R_1 + 2R_2} \begin{pmatrix} 2a-4d & 2b-4e & 2c-4f \\ a & b & c \\ g & h & i \end{pmatrix}.$$

(b) The vector v_1 from $(1,1)$ to $(3,5)$ is $v_1 = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$.

The vector v_2 from $(1,1)$ to $(4,-1)$ is $v_2 = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$. The triangle's area is

$$\frac{1}{2}\left|\det\begin{pmatrix} 2 & 3 \\ 4 & -2 \end{pmatrix}\right| = \frac{1}{2}\Big| -4-12 \Big| = 8.$$

(c) The area is

$$|\det(A)|\,\text{Area}(S) = \left|\det\begin{pmatrix} 3 & -1 \\ 1 & -2 \end{pmatrix}\right| \cdot 6 = \Big| -6+1 \Big| \cdot 6 = 30.$$

(d) (i) is true by a standard fact for determinants.
(ii) is false: if $\det(A) = 4$, then A is invertible and $\det(A^{-1}) = 1/4$.
(iii) is true: if $\det(A) = -2$ then $\det(A^4) = (\det A)^4 = (-2)^4 = 16$.
(iv) is true: if 0 is an eigenvalue of A, then

$$\det(AB) = \det(A)\det(B) = 0\det(B) = 0.$$

3. On this page, you do not need to show work and there is no partial credit. Parts (a) through (d) are unrelated.

(a) (2 points) Find $\det \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 2 & 7 & 10 & 1 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$.

○ 0 ○ 2 ○ 4 ○ -4 ○ 6

○ -6 ○ 12 ● -12 ○ none of these

(b) (2 points) Suppose A is an $n \times n$ matrix. Which **one** of the following statements is correct?

○ An eigenvector of A is a vector v such that $Av = \lambda v$ for a nonzero scalar λ.

● An eigenvector of A is a nonzero vector v such that $Av = \lambda v$ for a scalar λ.

○ An eigenvector of A is a nonzero scalar λ such that $Av = \lambda v$ for some vector v.

○ An eigenvector of A is a nonzero vector v such that $Av = \lambda v$ for a nonzero scalar λ.

(c) (2 points) Let A be the 2×2 matrix that reflects every vector $\begin{pmatrix} x \\ y \end{pmatrix}$ in $\mathbf{R}^2$ across the line $y = 4x$. Which **one** of the vectors below is in the (-1)-eigenspace of A?

○ $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$ ○ $\begin{pmatrix} 1 \\ -4 \end{pmatrix}$ ● $\begin{pmatrix} 4 \\ -1 \end{pmatrix}$ ○ $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$ ○ $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

(d) (4 points) Which of the following statements are true? Clearly fill in the bubble for all that apply.

○ If A is a diagonalizable 2×2 matrix, then A must have two different eigenvalues.

○ If $\lambda = 3$ is an eigenvalue of an $n \times n$ matrix A, then the columns of $A - 3I$ must be linearly independent.

● If A is an $n \times n$ matrix and every vector in $\mathbf{R}^n$ is in the 1-eigenspace of A, then $A = I_n$ (the $n \times n$ identity matrix).

● If A is a 4×4 matrix, then it is impossible for A to have 5 different eigenvalues.

Solution: Problem 3.

(a) Call our given matrix A. If we swap the last two rows of A, we get the matrix B below:

$$B = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 2 & 7 & 10 & 1 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}.$$

Since B is triangular we know $\det(B) = 1(2)(2)(3)(1) = 12$, so since B was obtained from A from a row swap, we know $\det(A) = -\det(B) = -12$.

Alternatively, we could have used a cofactor expansion, but the one-step solution above is significantly faster.

(b) This was copied and pasted from a supplemental problems PDF. The correct answer is nearly word-for-word the definition of eigenvector.

(c) For reflection across the line $y = 4x$, the (-1)-eigenspace is the line through the origin perpendicular to $y = 4x$, which is the line $y = -(1/4)x$. Exactly one vector among the answer choices that satisfies this, namely $\begin{pmatrix} 4 \\ -1 \end{pmatrix}$.

(d) (i) is not necessarily true: for example, $A = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$ is diagonalizable but only has one eigenvalue.

(ii) is not true: if $\lambda = 3$ is an eigenvalue of A, then the columns of $A - 3I$ must be linearly **dependent** since $A - 3I$ is not invertible.

(iii) is true: if every vector in $\mathbf{R}^n$ is in the 1-eigenspace of A, then $Ax = x$ for every x in $\mathbf{R}^n$, which means $A = I$.

Alternatively, we could note that since $e_1, \ldots, e_n$ are eigenvectors of A for eigenvalue $\lambda = 1$, we $A = CDC^{-1}$ where $C = \begin{pmatrix} e_1 & e_2 & \cdots & e_n \end{pmatrix} = I$ and D has 1 for every entry on the diagonal. In other words, $A = I\ I\ I^{-1} = I$.

(iv) is true: an $n \times n$ matrix can have a max of n different eigenvalues, so a 4×4 matrix can't have 5 different eigenvalues. One way of seeing this is that the characteristic polynomial of A has degree 4 and therefore has at most 4 different roots, so A has at most 4 different eigenvalues.

4. **See the next page for answers and solutions to this problem.**

(a) (3 points) Suppose A is a 6×6 matrix and its characteristic polynomial is

$$\det(A - \lambda I) = \lambda^2(5-\lambda)^3(2-\lambda).$$

Which of the following statements are true? Fill in the bubble for all that apply.

- ● If the null space of A is a line, then A cannot be diagonalizable.
- ● A cannot be a stochastic matrix.
- ● The solution set to the equation $Ax = 2x$ is a line.

(b) (2 pts) Suppose A be a 2×2 matrix whose trace is 15 and whose determinant is 50.

Enter the eigenvalues of A here: ________________.

(c) (2 pts) Write a 2×2 matrix A that is invertible but not diagonalizable.

$$A = \begin{pmatrix} & \\ & \end{pmatrix}$$

(d) (3 points) Write a 3×3 upper-triangular matrix B with exactly one eigenvalue $\lambda = -2$, with the property that the (-2)-eigenspace of B is a line. Write your answer below.

$$B = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$$

Solution: Problem 4.

(a) (i) is true: if the null space of A is a line, then the 0-eigenspace has algebraic multiplicity 2 (due to the "λ^2" term) but only geometric multiplicity 1, so the sum of geometric multiplicities for the 6×6 matrix will be at most 5, thus A cannot have 6 linearly independent eigenvectors.

(ii) is true: $\lambda = 1$ is an eigenvalue of every stochastic matrix, but this matrix has eigenvalues 0, 2, and 5.

(iii) is true: the eigenvalue $\lambda = 2$ has algebraic multiplicity 1 due to the factor of $2 - \lambda$ in the characteristic polynomial, so it automatically has geometric multiplicity 1.

(b) The characteristic polynomial is

$$\lambda^2 - \mathrm{Tr}(A)\lambda + \det(A) = \lambda^2 - 15\lambda + 50 = (\lambda - 5)(\lambda - 10),$$

so the eigenvalues are 5 and 10. Given the setup of the problem, you could also just come up with a matrix whose trace is 15 and determinant is 50 and see immediately that its eigenvalues are 5 and 10, for example

$$\begin{pmatrix} 5 & 0 \\ 0 & 10 \end{pmatrix}.$$

On this problem, if you put -5 and -10 as your answer, you would get 0 points out of 2 since a matrix with those eigenvalues would have a negative trace.

(c) Many examples possible.

- Any matrix with 2 non-real complex eigenvalues is invertible but not diagonalizable in the Math 1553 sense (since we require a basis of eigenvectors of $\mathbf{R}^n$ in order to be diagonalizable in 1553).
- Any invertible matrix with exactly one real eigenvalue, whose eigenvalue only has geometric multiplicity 1, will not be diagonalizable. For example,

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix}, \quad A = \begin{pmatrix} 5 & 10 \\ 0 & 5 \end{pmatrix}, \quad \text{etc.}$$

  Any upper-triangular or lower triangular matrix that has the same entries on the diagonal but a non-zero entry in the off-diagonal will work.

(d) We need the two following things.

- Since B is upper-triangular with exactly one real eigenvalue -2, we need each of its diagonal entries to be -2.
- Since the (-2)-eigenspace is a line, we need $B+2I$ to have **two pivots** so that there will be only one free variable for $\left(B+2I \mid 0\right)$.

Many answers are possible, for example

$$B = \begin{pmatrix} -2 & 1 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{pmatrix}, \quad B = \begin{pmatrix} -2 & 1 & 0 \\ 0 & -2 & 3 \\ 0 & 0 & -2 \end{pmatrix}, \quad B = \begin{pmatrix} -2 & 2 & 3 \\ 0 & -2 & 5 \\ 0 & 0 & -2 \end{pmatrix}.$$

Some **incorrect** answers are listed below:

$$B = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \quad B = \begin{pmatrix} -2 & 0 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \quad B = \begin{pmatrix} -2 & 0 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{pmatrix}.$$

All of those incorrect matrices above have a (-2)-eigenspace that is either $\mathbf{R}^3$ or is a plane.

5. Free response. Show your work unless otherwise indicated! A correct answer without appropriate work will receive little or no credit.

For this problem, let $A = \begin{pmatrix} 3 & 2 & -10 \\ 0 & 4 & -5 \\ 0 & 0 & 3 \end{pmatrix}$.

(a) (2 points) Write the eigenvalues of A. You do not need to show your work on this part.
A is upper-triangular, so its eigenvalues are its diagonal entries: $\lambda = 3$ and $\lambda = 4$.

(b) (5 points) For each eigenvalue of A, find a basis for the corresponding eigenspace.
For the 3-eigenspace:

$$(A - 3I|0) = \left(\begin{array}{ccc|c} 0 & 2 & -10 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 0 & 1 & -5 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{RREF} \left(\begin{array}{ccc|c} 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

We see x_1 and x_3 are free, and $x_2 = 5x_3$:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ 5x_3 \\ x_3 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 5 \\ 1 \end{pmatrix}. \quad \boxed{\text{Basis for 3-eigenspace}: \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 5 \\ 1 \end{pmatrix} \right\}}.$$

For the 4-eigenspace:

$$(A-4I|0) = \left(\begin{array}{ccc|c} -1 & 2 & -10 & 0 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & -2 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -2 & 10 & 0 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & -2 & 0 \end{array}\right) \xrightarrow{RREF} \left(\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

This gives $x_1 - 2x_2 = 0$, where x_2 is free and $x_3 = 0$. So $x_1 = 2x_2$, and we get

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2x_2 \\ x_2 \\ 0 \end{pmatrix} = x_2 \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}. \quad \boxed{\text{Basis for 4-eigenspace}: \left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \right\}}.$$

(c) (3 pts) The matrix A is diagonalizable. Write a 3×3 matrix C and a diagonal matrix D so that $A = CDC^{-1}$. You do not need to show your work on this part.

We form C using linearly independent eigenvectors and form D using the eigenvalues written **in the corresponding order**. Many answers are possible. For example,

$$C = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 5 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}.$$

or

$$C = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 0 & 5 \\ 0 & 0 & 1 \end{pmatrix}, \qquad D = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

6. Free response. Show your work! A correct answer without sufficient work may receive little or no credit.

(a) (6 points) Find the complex eigenvalues of the matrix $A = \begin{pmatrix} 5 & 9 \\ -2 & -1 \end{pmatrix}$. For the eigenvalue with **positive** imaginary part, find a corresponding eigenvector v. Simplify your eigenvalues as much as possible!

The eigenvalues are: $2+3i$ and $2-3i$ $\qquad v = \begin{pmatrix} -9 \\ 3-3i \end{pmatrix}$.

Solution: We solve for λ in the characteristic equation:

$$0 = \det(A - \lambda I) = \lambda^2 - \text{Tr}(A)\lambda + \det(A) = \lambda^2 - 4\lambda + 13,$$

$$\lambda = \frac{4 \pm \sqrt{(-4)^2 - 4(13)}}{2} = \frac{4 \pm \sqrt{-36}}{2} = \frac{4 \pm 6i}{2} = 2 \pm 3i.$$

We find v by using the 2×2 eigenvector trick for $\lambda = 2 + 3i$:

$$\left(A - (2+3i)I \mid 0 \right) = \left(\begin{array}{cc|c} 5-(2+3i) & 9 & 0 \\ (*) & (*) & 0 \end{array} \right) = \left(\begin{array}{cc|c} 3-3i & 9 & 0 \\ (*) & (*) & 0 \end{array} \right) = \left(\begin{array}{cc|c} a & b & 0 \\ (*) & (*) & 0 \end{array} \right).$$

An eigenvector is $v = \begin{pmatrix} -b \\ a \end{pmatrix} = \begin{pmatrix} -9 \\ 3-3i \end{pmatrix}$.

Other answers are possible, such as $v = \begin{pmatrix} 9 \\ -3+3i \end{pmatrix}$, or $\begin{pmatrix} 3+3i \\ -2 \end{pmatrix}$ or $\begin{pmatrix} -3-3i \\ 2 \end{pmatrix}$.

(b) (4 points) Find all values of c (if there are any) so that the matrix below has exactly one real eigenvalue with algebraic multiplicity 2:

$$\begin{pmatrix} 4 & 3 \\ c & -2 \end{pmatrix}.$$

Enter your answer here: ____________________

Solution: The characteristic polynomial for the matrix is

$$\lambda^2 - \text{Tr}(A)\lambda + \det(A) = \lambda^2 - 2\lambda + (-8 - 3c).$$

For this to be a perfect square, it must equal $(\lambda - 1)^2 = \lambda^2 - 2\lambda + 1$. Setting this equal to the above gives us

$$-8 - 3c = 1, \qquad 3c = -9, \qquad c = -3.$$

7. Free response. Show your work except on part (a) part (i)! A correct answer elsewhere without sufficient work will receive little or no credit. Parts (a) and (b) are unrelated.

(a) (7 points) The Great Journey is an online game where participants play either as the Bard or the Cleric. A total of 110 people play the game. This year, 80 players use the Bard and 30 players use the Cleric. Each year:

- 60% of players who use the Bard keep playing as the Bard, while 40% switch to the Cleric.
- 30% of players who use the Cleric keep playing as the Cleric, while 70% switch to the Bard.

(i) Write a positive stochastic matrix A and a vector x so that Ax will give the number of players for the Bard and the Cleric (in that order) next year. You do not need to compute Ax.

$$A = \begin{pmatrix} 0.6 & 0.7 \\ 0.4 & 0.3 \end{pmatrix}, \qquad x = \begin{pmatrix} 80 \\ 30 \end{pmatrix}.$$

(ii) In the long run, roughly how many yearly players will the Cleric have?

Enter your answer here: ____________.

Solution: $\left(A - I \mid 0\right) = \left(\begin{array}{cc|c} -0.4 & 0.7 & 0 \\ 0.4 & -0.7 & 0 \end{array}\right) \xrightarrow{RREF} \left(\begin{array}{cc|c} 1 & -7/4 & 0 \\ 0 & 0 & 0 \end{array}\right)$, so for the 1-eigenspace we get $x_1 = \frac{7}{4}x_2$ where x_2 is free, thus the vector $v = \begin{pmatrix} 7/4 \\ 1 \end{pmatrix}$ spans the 1-eigenspace. The steady state vector is

$$w = \frac{1}{7/4+1}\begin{pmatrix} 7/4 \\ 1 \end{pmatrix} = \frac{1}{11/4}\begin{pmatrix} 7/4 \\ 1 \end{pmatrix} = \begin{pmatrix} 7/11 \\ 4/11 \end{pmatrix}.$$

Therefore, over time, the Cleric will have roughly 4/11 of all players, which is $(4/11) * 110 = 40$.

(b) (3 points) Find $\det \begin{pmatrix} 1 & -2 & 3 & 0 \\ 0 & 1 & 3 & 0 \\ 2 & 1 & 4 & 2 \\ 0 & 2 & 1 & 0 \end{pmatrix}$. Enter your answer here: ____________.

Solution: Let's call the matrix A. We use the cofactor expansion along the fourth column:

$$\det(A) = 0 + 0 + 2C_{34} + 0 = 2(-1)^{3+4}\det(A_{34}) = -2\det\begin{pmatrix} 1 & -2 & 3 \\ 0 & 1 & 3 \\ 0 & 2 & 1 \end{pmatrix}$$
$$= -2\Big(1(1-6) - (-2)(0-0) + 3(0-0)\Big) = -2(-5) = 10.$$

This page is reserved ONLY for work that did not fit elsewhere on the exam.

If you use this page, please clearly indicate (on the problem's page and here) which problems you are doing.