Math 1553 Exam 3, Spring 2025

Name		GT ID	

Circle your instructor and lecture below. Be sure to circle the correct choice!

Jankowski (A+HP, 8:25-9:15 AM) Jankowski (C, 9:30-10:20 AM)

Al Ahmadieh (I, 2:00-2:50 PM) Al Ahmadieh (M, 3:30-4:20 PM)

Please read the following instructions carefully.

- Write your initials at the top of each page. The maximum score on this exam is 70 points, and you have 75 minutes to complete it. Each problem is worth 10 points.
- There are no calculators or aids of any kind (notes, text, etc.) allowed. Unless stated otherwise, **the entries of all matrices on the exam are real numbers**.
- As always, RREF means "reduced row echelon form." The "zero vector" in $\mathbf{R}^n$ is the vector in $\mathbf{R}^n$ whose entries are all zero.
- On free response problems, show your work, unless instructed otherwise. A correct answer without appropriate work may receive little or no credit!
- We will hand out loose scrap paper, but it **will not be graded** under any circumstances. All answers and work must be written on the exam itself, with no exceptions.
- This exam is double-sided. You should have more than enough space to do every problem on the exam, but if you need extra space, you may use the *back side of the very last page of the exam*. If you do this, you must clearly indicate it.
- You may cite any theorem proved in class or in the sections we covered in the text.
- For questions with bubbles, either fill in the bubble completely or leave it blank. **Do not** mark any bubble with "X" or "/" or any such intermediate marking. Anything other than a blank or filled bubble may result in a 0 on the problem, and regrade requests may be rejected without consideration.

I, the undersigned, hereby affirm that I will not share the contents of this exam with anyone. Furthermore, I have not received inappropriate assistance in the midst of nor prior to taking this exam. I will not discuss this exam with anyone in any form until after 7:45 PM on Wednesday, April 9.

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1. TRUE or FALSE. Clearly fill in the bubble for your answer. If the statement is *ever* false, fill in the bubble for False. You do not need to show any work, and there is no partial credit. Each question is worth 2 points.

   (a) If A is a 3×3 matrix, then $\det(2A) = 8\det(A)$.

   ○ True

   ○ False

   (b) If A is a 3×3 matrix and $A \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, then $\det(A) = 0$.

   ○ True

   ○ False

   (c) If A is a 3×3 matrix and

   $$A \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ 12 \end{pmatrix},$$

   then the equation $Ax = 4x$ must have infinitely many solutions.

   ○ True

   ○ False

   (d) If u and v are eigenvectors of an $n \times n$ matrix A, then $u + v$ must also be an eigenvector of A.

   ○ True

   ○ False

   (e) If A is a 3×3 matrix with eigenvalues $\lambda_1 = 0$, $\lambda_2 = 1$, and $\lambda_3 = -4$, then A must be diagonalizable.

   ○ True

   ○ False

2. On this page, you do not need to show work and there is no partial credit. Parts (a) through (d) are unrelated.

(a) (2 points) Suppose $\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = 1$. Find $\det\begin{pmatrix} 2a-4d & 2b-4e & 2c-4f \\ a & b & c \\ g & h & i \end{pmatrix}$.

○ 1 ○ −1 ○ 2 ○ −2 ○ 6

○ 4 ○ −4 ○ 8 ○ −8 ○ none of these

(b) (2 points) Find the area of the triangle with vertices

$$(1,1), \quad (3,5), \quad (4,-1).$$

○ 1/2 ○ 1 ○ 2 ○ 4 ○ 5 ○ 8

○ 10 ○ 16 ○ 64 ○ none of these

(c) (2 points) Let T be the linear transformation $T(x,y) = (3x-y, x-2y)$, and let S be a rectangle in $\mathbf{R}^2$ with area 6. Find the area of $T(S)$.

○ 5 ○ 6 ○ 7 ○ 30 ○ 36

○ 42 ○ 48 ○ 72 ○ none of these

(d) (4 points) Suppose A and B are $n \times n$ matrices. Which of the following statements are true? Clearly fill in the bubble for all that apply.

○ $\det(AB) = \det(A)\det(B)$.

○ If $\det(A) = 4$, then $\det(A^{-1}) = -4$.

○ If $\det(A) = -2$, then $\det(A^4) = 16$.

○ If 0 is an eigenvalue of A, then $\det(AB) = 0$.

3. On this page, you do not need to show work and there is no partial credit. Parts (a) through (d) are unrelated.

(a) (2 points) Find $\det \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 2 & 7 & 10 & 1 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$.

○ 0 ○ 2 ○ 4 ○ -4 ○ 6

○ -6 ○ 12 ○ -12 ○ none of these

(b) (2 points) Suppose A is an $n \times n$ matrix. Which **one** of the following statements is correct?

○ An eigenvector of A is a vector v such that $Av = \lambda v$ for a nonzero scalar λ.

○ An eigenvector of A is a nonzero vector v such that $Av = \lambda v$ for a scalar λ.

○ An eigenvector of A is a nonzero scalar λ such that $Av = \lambda v$ for some vector v.

○ An eigenvector of A is a nonzero vector v such that $Av = \lambda v$ for a nonzero scalar λ.

(c) (2 points) Let A be the 2×2 matrix that reflects every vector $\begin{pmatrix} x \\ y \end{pmatrix}$ in $\mathbf{R}^2$ across the line $y = 4x$. Which **one** of the vectors below is in the (-1)-eigenspace of A?

○ $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$ ○ $\begin{pmatrix} 1 \\ -4 \end{pmatrix}$ ○ $\begin{pmatrix} 4 \\ -1 \end{pmatrix}$ ○ $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$ ○ $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

(d) (4 points) Which of the following statements are true? Clearly fill in the bubble for all that apply.

○ If A is a diagonalizable 2×2 matrix, then A must have two different eigenvalues.

○ If $\lambda = 3$ is an eigenvalue of an $n \times n$ matrix A, then the columns of $A - 3I$ must be linearly independent.

○ If A is an $n \times n$ matrix and every vector in $\mathbf{R}^n$ is in the 1-eigenspace of A, then $A = I_n$ (the $n \times n$ identity matrix).

○ If A is a 4×4 matrix, then it is impossible for A to have 5 different eigenvalues.

4. On this page, you do not need to show your work. Only your answers are graded. Parts (a)-(d) are unrelated.

   (a) (3 points) Suppose A is a 6×6 matrix and its characteristic polynomial is

   $$\det(A - \lambda I) = \lambda^2(5 - \lambda)^3(2 - \lambda).$$

   Which of the following statements are true? Fill in the bubble for all that apply.

   ○ If the null space of A is a line, then A cannot be diagonalizable.

   ○ A cannot be a stochastic matrix.

   ○ The solution set to the equation $Ax = 2x$ is a line.

   (b) (2 pts) Suppose A be a 2×2 matrix whose trace is 15 and whose determinant is 50.

   Enter the eigenvalues of A here: ________________.

   (c) (2 pts) Write a 2×2 matrix A that is invertible but not diagonalizable.

   $$A = \begin{pmatrix} & \\ & \end{pmatrix}$$

   (d) (3 points) Write a 3×3 upper-triangular matrix B with exactly one eigenvalue $\lambda = -2$, with the property that the (-2)-eigenspace of B is a line. Write your answer below.

   $$B = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$$

5. Free response. Show your work unless otherwise indicated! A correct answer without appropriate work will receive little or no credit.

For this problem, let $A = \begin{pmatrix} 3 & 2 & -10 \\ 0 & 4 & -5 \\ 0 & 0 & 3 \end{pmatrix}$.

(a) (2 points) Write the eigenvalues of A. You do not need to show your work on this part.

(b) (5 points) For each eigenvalue of A, find a basis for the corresponding eigenspace.

(c) (3 pts) The matrix A is diagonalizable. Write a 3×3 matrix C and a diagonal matrix D so that $A = CDC^{-1}$. You do not need to show your work on this part.

$C = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$ $D = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$

6. Free response. Show your work! A correct answer without sufficient work may receive little or no credit.

   (a) (6 points) Find the complex eigenvalues of the matrix $A = \begin{pmatrix} 5 & 9 \\ -2 & -1 \end{pmatrix}$. For the eigenvalue with **positive** imaginary part, find a corresponding eigenvector v. Simplify your eigenvalues as much as possible!

   The eigenvalues are: ____________________ $v = \begin{pmatrix} \\ \\ \end{pmatrix}$.

   (b) (4 points) Find all values of c (if there are any) so that the matrix below has exactly one real eigenvalue with algebraic multiplicity 2:

   $\begin{pmatrix} 4 & 3 \\ c & -2 \end{pmatrix}$.

   Enter your answer here: ____________________

7. Free response. Show your work except on part (a) part (i)! A correct answer elsewhere without sufficient work will receive little or no credit. Parts (a) and (b) are unrelated.

   (a) (7 points) The Great Journey is an online game where participants play either as the Bard or the Cleric. A total of 110 people play the game. This year, 80 players use the Bard and 30 players use the Cleric. Each year:

   - 60% of players who use the Bard keep playing as the Bard, while 40% switch to the Cleric.
   - 30% of players who use the Cleric keep playing as the Cleric, while 70% switch to the Bard.

   (i) Write a positive stochastic matrix A and a vector x so that Ax will give the number of players for the Bard and the Cleric (in that order) next year. You do not need to compute Ax.

$$A = \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix} \qquad x = \begin{pmatrix} \quad \\ \quad \end{pmatrix}$$

   (ii) In the long run, roughly how many yearly players will the Cleric have?

   Enter your answer here: ____________.

   (b) (3 points) Find $\det \begin{pmatrix} 1 & -2 & 3 & 0 \\ 0 & 1 & 3 & 0 \\ 2 & 1 & 4 & 2 \\ 0 & 2 & 1 & 0 \end{pmatrix}$. Enter your answer here: ____________.

This page is reserved ONLY for work that did not fit elsewhere on the exam.

If you use this page, please clearly indicate (on the problem's page and here) which problems you are doing.